OAA OMC
Orange Math Competitions

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OMC 10

Orange Mathematics Competitions
Saturday, January 23, 2021

INSTRUCTIONS

1. DO NOT LOOK AT THE PROBLEMS UNTIL YOU ARE READY TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/` and mark you answer to each problem on the AMC 10 Answer Sheet with a number 2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the OMC, **you must submit your answers using the Submission Form found at** `https://tinyurl.com/omc10submission`**. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only pencils, erasers, rulers, and scratch paper are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10/12 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/`. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, sign your name in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-1012/`.
9. Enjoy the problems!

The Committee on the Orange Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

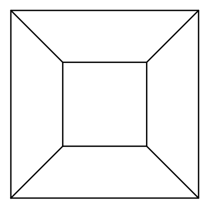
1. What is the value of $-7(-5(-4(1-2)-3)-6)-8$?

   (A) -393 **(B)** -85 **(C)** 69 **(D)** 98 **(E)** 406

2. 250 students have registered for SFA's OMC 10/12. A proctor is allowed to watch over at most ten students to ensure no foul play occurs. If 70% of the registrants actually show up to the contest, what is the minimum number of proctors needed?

   (A) 17 **(B)** 18 **(C)** 25 **(D)** 35 **(E)** 36

3. In the figure below, the two squares share a center. If the outer square has side length $\sqrt{5}$ and the inner square has side length 1, what is the area of one of the four congruent trapezoids inside the outer square but outside the inner square?



   (A) $\frac{4}{5}$ **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** $\frac{3}{2}$ **(E)** 2

4. Janice, Ryan, and Samantha all roll a fair six sided die labeled $1, 2, \cdots, 6$. Given that they all role different numbers, what is the probability Ryan's number will be the largest?

   (A) $\frac{1}{6}$ **(B)** $\frac{55}{216}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

5. A sequence is defined by $a_0 = \frac{1}{2025}$ and
$$a_{n+1} = a_n + a_n^2 + a_n^3 + \cdots .$$
   What is the value of a_{2021}?

   (A) $\frac{1}{7}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

6. Let $ABCD$ be a trapezoid with $\angle B = 90^\circ$ and $AD = 2 \cdot BC$. What is the sum of all possible angles $\angle BAD$?

   (A) 90° **(B)** 120° **(C)** 210° **(D)** 240° **(E)** 270°

7. Ben chooses three vertices of a regular hexagon at random and draws the triangle with vertices at these three points. What is the probability that the area of this triangle is at least one-third of the area of the hexagon?

   (A) $\frac{1}{10}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{7}{10}$ **(E)** $\frac{9}{10}$

8. A positive integer has 16 factors, k of which are even. What is the sum of all possible values of k?

   (A) 15 **(B)** 31 **(C)** 34 **(D)** 49 **(E)** 65

9. John has a deck of 2020 cards numbered 1 through 2020 in that order, with the bottom card labeled 1 and the top card labeled 2020. At each step, he removes the top and bottom cards from the deck and places both cards, in that order, at the top of the remaining stack. For example, if the top and bottom cards in the deck were labeled A and B, respectively, then after one step, the top two cards in the deck would be A and B, in that order. After 2100 steps, what is the number on the second card?

   (A) 79 **(B)** 80 **(C)** 81 **(D)** 1939 **(E)** 1940

10. Let a_n be a strictly increasing sequence of positive integers defined for all integers $n \geq 1$ such that the sum of any 2020 consecutive terms is divisible by 2020. If $a_i = i$ for all integers $1 \leq i \leq 2019$, what is the smallest possible value of a_{4040}?

    (A) 4040 **(B)** 5050 **(C)** 6060 **(D)** 7070 **(E)** 8080

11. Olivia writes down the binary representation for all positive integers $1 \leq n \leq 1024$. How many more 1s does Olivia write than 0s?

    (A) 1013 **(B)** 1014 **(C)** 1015 **(D)** 1023 **(E)** 1024

12. Let $ABCD$ be a rectangle with $AB = 13$ and $BC = 5$. Curtis folds rectangle $ABCD$ along a line ℓ passing through A such that point B lies on segment CD. He notices that ℓ intersects segment BC at a point X. What is the length of BX?

    (A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{13}{5}$ **(E)** 3

13. Let $Q(x)$ be a quadratic with leading coefficient one, real coefficients, two real roots, sum of roots s, and product of roots p. If $s = p$, how many of the following **must** be true?

    I: The linear and constant coefficient are equal.

    II: For every positive real number a, there exists a polynomial $Q(x)$ that satisfies the given conditions and has $p = a$.

    III: For every positive real number d, there exists exactly two polynomials $Q(x)$ that satisfies the given conditions and has the positive difference of the two roots equal to d.

    IV: For every positive integer k, there exists a polynomial $Q(x)$ that satisfies the given conditions and has at least one root equal to k.

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

14. Three spheres of radius 1 are all externally tangent to each other and externally tangent to a plane P. There exists an unique sphere S such that S is tangent to P and all three spheres of radius 1. Find the radius of S.

    (A) $\frac{\sqrt{3}}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{\sqrt{3}}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

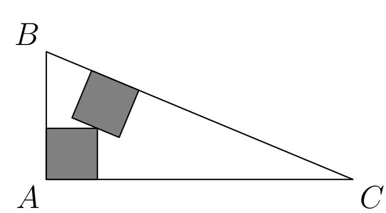
15. Let $f(n)$ be the sum of all positive divisors of n. The smallest interval containing all possible values of $\frac{f(12n)}{f(n)}$ can be expressed as $(a, b]$. What is the value of $b - a$?

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

16. Define $P(x)$ to be a degree 40 polynomial with real coefficients such that for all real x, $xP(x+40) = (x+40)P(x+39)$. If $P(-1) = 1$, find $P(42)$.

    (A) -1772 **(B)** -861 **(C)** 42 **(D)** 861 **(E)** 1722

17. In right triangle ABC, two identical squares are placed as shown below. If the perimeter of one of the squares is exactly one-tenth of the perimeter of ABC, what is the ratio of the shaded area to the area of ABC?



    (A) $\frac{1}{20}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{5}$

18. In square $ABCD$ of side length 1, a point P is selected inside the square. Let R be the region that the interior of $ABCD$ sweeps out when rotating about the point P. Let S be the set of points P such that the area of R is π. What is the area enclosed by S?

(A) $\frac{\pi}{3} - \sqrt{3} + 1$ **(B)** $\frac{\pi}{2} - \frac{\pi\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + 1$ **(C)** $\frac{\pi}{8}$ **(D)** $\pi - \frac{\pi\sqrt{3}}{2}$ **(E)** $\frac{\sqrt{3}}{4}$

19. How many ordered triples (a, b, c) of positive integers less than or equal to 20 are there such that $\frac{a-b}{c}$, $\frac{b-c}{a}$, and $\frac{c-a}{b}$ are all integers?

(A) 20 **(B)** 66 **(C)** 101 **(D)** 158 **(E)** 198

20. Let ABC be a right triangle with at least 2 integer sides. Let A', B', and C' be the reflections of A across BC, B across CA, and C across AB respectively. How many noncongruent triangles ABC are there such that triangle $A'B'C'$ has area 2025?

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 32

21. Define the function $f\left(\frac{m}{n}\right)$ to be the numerator of the fraction $\frac{m}{n}$ when put in simplest form. For example, $f\left(\frac{6}{8}\right) = 3$ and $f\left(\frac{2}{5}\right) = 2$. What are the last two digits of the sum

$$f\left(\frac{1}{2^{2020}}\right) + f\left(\frac{2}{2^{2020}}\right) + f\left(\frac{3}{2^{2020}}\right) + \cdots + f\left(\frac{2^{2020}-1}{2^{2020}}\right)?$$

(A) 01 **(B)** 05 **(C)** 25 **(D)** 27 **(E)** 75

22. There are 8 students in a classroom, in which friendship is mutual. Suppose among any three students, there is an odd number of friend pairs. How many possible ways can the students be friends with each other?

(A) 24 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 1024

23. 100 students in the Arvine Unified School District are taking a quiz. Each student randomly submits a real number between 0 and 1. All students submit a different real number. A student's score is the minimum positive difference between his or her number and another student's number. Let M be the maximum number of distinct scores. What is the probability that there will be M distinct scores?

(A) $\frac{2^{98}}{100!}$ **(B)** $\frac{2^{99}}{100!}$ **(C)** $\frac{2^{98}}{99!}$ **(D)** $\frac{2^{99}}{99!}$ **(E)** $\frac{2^{98}}{98!}$

24. Let ABC be a triangle with $AB = 8$, $BC = 7$ and $CA = 5$. Let ω denote the circumcircle of ABC. The tangent to the ω at B meet line AC and the tangent at C to ω at points D and E, respectively. Let the circumcircle of ABE meet segment CD at F. What is the length of EF?

(A) $\frac{35}{8}$ **(B)** $\frac{7\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{645}}{5}$ **(D)** $\frac{5\sqrt{70}}{8}$ **(E)** $\frac{40}{7}$

25. Given that $x^4 + ax^3 + bx^2 + 4ax + 16$ has four distinct positive real roots for integers a and b, what is the smallest possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 23 **(E)** 26

2021

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DO NOT OPEN UNTIL SATURDAY, January 23, 2021

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Correspondence about the problems and solutions
for this exam should be sent by email to:

ocmathcircle@gmail.com.

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent OMC 10 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE January 23, 2021.

2. YOU must not verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on maa.org/amc under "AMC 10/12") that you followed all rules associated with the administration of the exam.

3. If you chose to obtain an AMC 10/12 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedEx or UPS is strongly recommended.

4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, World Wide Web, or digital media of any type during this period is a violation of the competition rules.

The **Orange Math Competitions**
are made possible by the contributions of
the following problem-writers, test-solvers,
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