Team Round

October 23rd, 2021

1. **[2]** A triple of positive integers (x, y, z) is said to be a Pythagorean triple if $x^2 + y^2 = z^2$. We call a Pythagorean triple, an inflatable triple if (x^2, y^2, z^2) is also a Pythagorean triple. Find the number of inflatable triples.

2. **[2]** Let a, b, c, d, e, f be a random permutation of the integers from 1 to 6. If the probability $ab+cd+ef$ is even can be expressed in simplest form as $\frac{m}{n}$, find $m + n$.

3. **[3]** Find the sum of all positive multiples of 17 less than 200 that can be expressed as the sum of two squares.

4. **[3]** Suppose the sequence a_1, a_2, a_3,... satisfies

$$a_n = a_{n-1} + 2a_{n-2} + 4a_{n-3} + \cdots + 2^{n-2}a_1$$

for all $n \geq 2$, and $a_1 = 2021$. Find the last two digits of a_{2022}.

5. **[5]** Let I and G be the incenter and centroid of scalene $\triangle ABC$, respectively. If $IG||BC$, then $\frac{AB}{AC} > c$ for some constant c. What is the smallest possible value of $\lfloor \frac{1}{c} \rfloor$?

6. **[5]** An ant starts at $(0, 0)$ on the coordinate plane, and takes a path moving only one unit upwards or one unit to the right each step to $(3, 3)$. Once reaching $(3, 3)$, it takes a path moving only one unit downwards or one unit to the left each step back to $(0, 0)$, such that this second path does not intersect the first path except at $(0, 0)$. How many ways can this be done?

7. **[6]** Let a, b, c be the three distinct roots of x^3+4x^2+4x+5. What is $(a^2+a+1)(b^2+b+1)(c^2+c+1)$?

8. **[7]** Kodvick and Broadwick are playing a dice game, where they alternate rolling a fair die. If at any point any player rolls a 1, that player instantly loses. Kodvick begins by rolling a 6-sided die labeled from 1 to 6. Suppose he rolls some $j > 1$ (if $j = 1$, he loses). Then Broadwick must roll a j-sided die labeled from 1 to j, and the process alternates until someone rolls a 1. If the expected number of rolls it takes before one player loses is $\frac{m}{n}$, for relatively prime positive integers m and n, compute $m + n$.

9. **[8]** Given that $\frac{1}{47}$ is a repeating decimal that has period 46, what is the sum of those 46 digits?

10. **[9]** Let ABC be a triangle with incenter I and incircle ω. Let T be the foot of the altitude from I onto AC and let S be the reflection of A over T. Line SI intersects line AB at X. Given $AB = 20$, $AC = 23$, $AX = 15$ and $\angle BAC < 120°$, the sum of all possible values of BC can be expressed as $\frac{m}{n}$ for relatively prime numbers m and n. Compute $m + n$.