Round of Choices

October 23rd, 2021

Set 1

1. Let $P(x)$ be a monic nonconstant polynomial. Suppose that for any root r of $P(x)$, $-(2020 - 21r)$ is also a root. Find largest possible value of $|P(89)|$ less than 2021.
2. The integers from 1 to 100 are written on a board. Every second, Timothy changes every number n on the board to $\phi(n + 4)$, where ϕ is the Euler Totient Function. After a year, what is the sum of the numbers on the board?
3. Suppose we have 10 points equally spaced on a circle. We would like to color them red or blue. How many ways are there to color 7 points red and 3 points blue so that no two blue points are next to each other?

Set 2

1. Define a special number as a positive integer n for which each number from 1 to n can be expressed as the sum of distinct divisors of n. Suppose $2021^{2021} \cdot m$ is a special number. Find the smallest possible integer value of m.
2. A particle is located at $(0, 0)$, initially facing the $+x$ direction. A move consists of moving 5 in the direction the particle is facing and then rotating the particle 45° counterclockwise. What will the greatest integer less than the product of its coordinates be after the particle makes 50 of these moves?
3. In $\triangle ABC$, let I be the incenter and O the circumcenter. Let AI meet the circumcircle of $\triangle ABC$ again at M. Suppose $IM = MO$. If $AC = 20$ and $AB = 22$, then BC can be expressed as $\sqrt{n}$ for a positive integer n. Find n.

Set 3

1. How many ordered triples (a, b, c) of positive integers less than or equal to 10 are there such that $2^a + 2^b + 2^c$ is a power of 2?
2. Suppose two perfect squares differ by 23. Find the larger of the two squares.
3. Define the functions $f_0, f_1, f_2 \ldots : \mathbb{N} \to \mathbb{R}$ via $f_p(n) = \lfloor \sqrt[p]{n} \rfloor - \lfloor \sqrt[p]{n-1} \rfloor$. Compute

$$\sum_{p=1}^{100} \sum_{n=1}^{100} f_p(n).$$

Set 4

1. Suppose $f(x) = x^3 + 20x^2 - 20x + 1$. Given that $f(10+i) = 2751 + 679i$, find the real part of $f(10-i)$.

2. How many positive integer divisors of $N = 2^9 3^{10} 5^{11}$ leave a remainder of 1 when divided by 4?

3. An equilateral triangle ABC of side length 1 has center O. A point P is selected on side AB such that $\frac{BP}{AP} = 3$. Let Q be a point on AC such that $OP = OQ$. The maximum possible area of quadrilateral $APOQ$ can be expressed as $\frac{\sqrt{m}}{n}$ for positive integers m and n where m is square-free. Compute $m + n$.

Set 5

1. A very bouncy ball with negligible size is shot from the top left hand corner of a unit square. At the instant the ball is fired, the angle the ball makes with the top of the unit square is θ. If the ball rebounds off the sides 2020 times before reaching a corner of the square, find the number of different possible values for θ.

2. Suppose for a positive integer n that
$$\frac{\cos 2021x}{\cos x} = \frac{\sin 2022x}{\sin 2x} - \frac{\sin nx}{\sin 2x}$$
for all x such that both sides are defined. Find n.

3. For how many ordered pairs (a, b) are there such that $102a(a^2 + 169) = b^4$?