OMC 12 Solutions

STEAM for All

January 2021

1 Answer Key

1. E
2. B
3. C
4. E
5. D
6. E
7. C
8. D
9. D
10. D
11. B
12. D
13. A
14. B
15. C
16. D
17. C
18. A
19. C
20. A
21. C
22. A
23. E
24. D
25. C

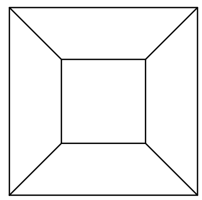
2 Solutions

1. If 15% of x is equal to 60% of y, what is the value of $\frac{x}{y}$?

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

   Solution: We are given $0.15x = 0.6y$. This means that $\frac{x}{y} = \frac{0.6}{0.15} = 4 \rightarrow \boxed{E}$.

2. In the figure below, the two squares share a center. If the outer square has side length $\sqrt{5}$ and the inner square has side length 1, what is the area of one of the four congruent trapezoids inside the outer square but outside the inner square?



   (A) $\frac{4}{5}$ **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** $\frac{3}{2}$ **(E)** 2

   Solution: The area of one of the congruent trapezoids must be one-fourth of the area between the two squares. The area between the squares is $\sqrt{5}^2 - 1^2 = 4$, so the area of one of the trapezoids is $\frac{1}{4} \cdot 4 = 1 \rightarrow \boxed{B}$.

3. Madison writes down the numbers 24 and 36 on a whiteboard. Every second, she replaces the two numbers on the board with their greatest common divisor and their least common multiple. After 2021 seconds, what is the sum of the two numbers on the whiteboard?

   (A) 60 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

   Solution: After the first second Madison has the numbers 12 and 72 on the board. Since 12 divides 72, beyond this point, the numbers Madison has on the board must be 12 and 72. It follows that the answer is $72 + 12 = 84$, or $\boxed{C}$.

4. There are N people in a circular pool of diameter 12 feet. To follow social distancing rules, no two people are less than 6 feet from each other. What is the maximum possible value of N?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

   Solution: Imagine a hexagon inscribed in the pool. Assign one person to each vertex of the hexagon and one person to the center. Under this arrangement, no two people are closer than 6 feet. Hence, since 7 is the largest possible answer choice, the answer is $\boxed{E}$.

5. Ben chooses three vertices of a regular hexagon at random and draws the triangle with vertices at these three points. What is the probability that the area of this triangle is at least one-third of the area of the hexagon?

   (A) $\frac{1}{10}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{7}{10}$ **(E)** $\frac{9}{10}$

   Solution: Without loss of generality, assume the side length of the hexagon is 1. Then, the area is equal to $\frac{3\sqrt{3}}{2}$, so we want all cases where the area of the triangle is greater than or equal to $\dfrac{\sqrt{3}}{2}$. There are three noncongruent triangles that Ben may have formed.

Case 1 (Three consecutive vertices): Here, there are 6 different ways to select these vertices (just choose which vertex of the six will be in the middle). In this case, the triangle is a 30-30-120 triangle, and thus is equal to two 30-60-90 triangles put together. Therefore, the area of one of these triangles is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{2}$

Case 2 (Two adjacent vertices and one vertex not adjacent to either of the other two): Here, there are 6 ways to select the two adjacent vertices and 2 ways to select the other. This is a right triangle, so its area is equal to $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2} \geq \frac{\sqrt{3}}{2}$.

Case 3 (Every other vertex): There are only 2 cases here. The area of the equilateral triangle is just $\frac{\sqrt{3}}{4}(\sqrt{3})^2 = \frac{3\sqrt{3}}{4} \geq \frac{\sqrt{3}}{2}$ because the side length of the triangle is $\sqrt{3}$.

Thus, in cases 2 and 3 our property is satisfied so our probability is $\dfrac{12+2}{20} = \dfrac{7}{10} \to \boxed{D}$.

6. Define $p(r)$ to be the number of lattice points in the region enclosed by $x^2 + y^2 \leq r$. For which positive integers r is $p(r)$ odd?

(A) Odd perfect squares **(B)** Even perfect squares
(C) All perfect squares **(D)** Non-squares **(E)** All positive integers

Solution: Note that if a point is inside the region, reflecting it over the origin gives another point that is in the region. We can pair up all points in this way except for the origin, so the $p(r)$ is odd for All positive integers.$\to \boxed{E}$

7. How many real solutions are there to the equation

$$\log_2(\log_4(x^2)) = \log_4(\log_2(x^2))?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: Take 4 to the power of both sides of the equation. Note that for every positive solution, there exists a negative solution as well (and $x = 0$ doesn't work). From here, we will first assume x is positive and then account for the negative values. We obtain

$$(\log_4(x^2))^2 = \log_2(x^2) \implies (\log_2(x))^2 = 2\log_2(x).$$

Therefore, $\log_2(x) \in \{0, 2\}$ Clearly $x = 1$ fails because of domain issues, so only $x = 4$ works. Now accounting for the negative solution $x = -4$, we reach an answer of $2 \to \boxed{C}$.

8. A positive integer has 16 factors, k of which are even. What is the sum of all possible values of k?

(A) 15 **(B)** 31 **(C)** 34 **(D)** 49 **(E)** 65

Solution: Note that the basic condition is that the prime factorization has 2^n where $n+1$ divides 16. Among the divisors, we must have exactly $\frac{n}{n+1}$ even, because that is how many have a nonzero power of 2. Since $n+1$ can be $1, 2, 4, 8, 16$, we know $16 \cdot \frac{n}{n+1}$ can be $0, 8, 12, 14, 15$, summing to 49. $\to \boxed{D}$

9. Let $f(n)$ denote the product of all the divisors of n. For how many positive integers $2 \leq n \leq 1000$ is $\log_n(f(n))$ an integer?

(A) 30 **(B)** 31 **(C)** 968 **(D)** 969 **(E)** 970

Solution: Note that when n is not a perfect square, all the divisors of n pair up to multiply to n, so $f(n)$ is a power of n and $\log_n f(n)$ is an integer. However, when n is a square, everything pairs up except the square root. It follows that $f(n)$ is not a power of n and $\log_n f(n)$ is not an integer. Thus, we subtract from 999 the number of squares from 2 to 1000. This is clearly 30 because the smallest is 2^2 and the largest is 31^2, so our answer is $999 - 30 = 969. \rightarrow \boxed{D}$.

10. Let $ABCD$ be a rectangle with $AB = 13$ and $BC = 5$. Curtis folds rectangle $ABCD$ along a line ℓ passing through A such that point B lies on segment CD. He notices that ℓ intersects segment BC at a point X. What is the length of BX?

(A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{13}{5}$ **(E)** 3

Solution: Let B fold to Y. Then, $AY = AB = 13$. Thus, $\triangle ADY$ is a 5-12-13 triangle. Specifically, $DY = 12$ and $YC = 1$. Now, our claim is that $\triangle ABX \sim \triangle BCY$. First, note that $BY \perp AX$, so $\angle YBC = 90^\circ - \angle ABY = \angle XAB$. Since $\triangle ABX$ and $\triangle BCY$ are both right triangles, it follows that they are similar. Thus, $BX = AB \cdot \frac{CY}{BC} = \frac{13 \cdot 1}{5} = \frac{13}{5}$ and the answer is $\boxed{D}$.

11. Let $Q(x)$ be a quadratic with leading coefficient one, real coefficients, two real roots, sum of roots s, and product of roots p. If $s = p$, how many of the following **must** be true?

I: The linear and constant coefficient are equal.

II: For every positive real number a, there exists a polynomial $Q(x)$ that satisfies the given conditions and has $p = a$.

III: For every positive real number d, there exists exactly two polynomials $Q(x)$ that satisfies the given conditions and has the positive difference of the two roots equal to d.

IV: For every positive integer k, there exists a polynomial $Q(x)$ that satisfies the given conditions and has at least one root equal to k.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: From the conditions, we can figure out that $Q(x) = x^2 - sx + s$. Because both roots are real, we must also have $s^2 - 4s$, the discriminant, greater than 0, so $s > 4$ or $s < 0$. Let's get rid of the obvious ones: I and II. Next, we note that the for III, the positive difference is exactly the square root of the discriminant of $s^2 - 4s$. It is clear that $s^2 - 4s$ can be any positive real number, so there always exists a s for which $d = \sqrt{s^2 - 4s}$ and III is true. As for IV, note that $Q(1) = 1$ no matter what, so $k = 1$ doesn't work. Thus, only III must be true, and our answer is $\boxed{B}$.

12. Three spheres of radius 1 are all externally tangent to each other and externally tangent to a plane P. There exists an unique sphere S such that S is tangent to P and all three spheres of radius 1. Find the radius of S.

(A) $\frac{\sqrt{3}}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{\sqrt{3}}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

Solution: Let O_1, O_2, O_3 be the centers of the larger spheres and let X be the center of the smaller sphere that is tangent to the plane and the three spheres and let r be the radius of this sphere. Connect all four centers with each other. Let Y be the foot of point X onto triangle $O_1O_2O_3$. Note that O_1XY is a right triangle with hypotenuse O_1X. We know that $O_1X = r + 1$ and $XY = 1 - r$. To find O_1Y, we note that $O_1O_2O_3$ is an equilateral triangle with center Y, so O_1Y is $\frac{\sqrt{3}}{3} \cdot O_1O_2 = \frac{2\sqrt{3}}{3}$ (by 30-60-90 triangle properties). Thus,

$$(1-r)^2 + \frac{4}{3} = (1+r)^2$$

$$\frac{4}{3} = 4r$$

This means $r = \frac{1}{3} \to \boxed{D}$.

13. Let x_1 and x_2 be two real numbers, each randomly selected in the interval $(0,1)$. What is the probability that $\lfloor \log_3(x_1) \rfloor + \lfloor \log_3(x_2) \rfloor$ is a multiple of 3?

(A) $\frac{55}{169}$ **(B)** $\frac{57}{169}$ **(C)** $\frac{5}{13}$ **(D)** $\frac{7}{13}$ **(E)** $\frac{55}{144}$

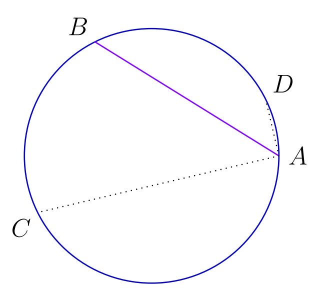
Solution: Note that the probability a randomly chosen number x from $(0,1)$ has $\lfloor \log_3(x) \rfloor = -k$ for a positive integer k is the probability x lies between $\frac{1}{3^k}$ and $\frac{1}{3^{k-1}}$, which is a probability of $\frac{2}{3^k}$. Thus, the probability that $-\lfloor \log_3(x) \rfloor$ leaves a remainder of 1 upon division by 3 is $\frac{2}{3} + \frac{2}{3^4} + \ldots = \frac{\frac{2}{3}}{1-\frac{1}{27}} = \frac{9}{13}$. Note that the probability it leaves a remainder of 1 upon division by 3 is exactly a third of that (so $\frac{3}{13}$) because we basically multiply the denominators by 3. Similarly, for when it leaves a remainder of 0 upon division by 3, we have $\frac{1}{13}$.

Now, $\lfloor \log_3(x_1) \rfloor + \lfloor \log_3(x_2) \rfloor$ is divisible by 3 if and only if $-\lfloor \log_3(x_1) \rfloor - \lfloor \log_3(x_2) \rfloor$ is, and that happens if the remainders pair up. So basically, we want 1 and 2, 2 and 1, or 0 and 0. That gives a probability of $\frac{9 \cdot 3}{13^2} + \frac{3 \cdot 9}{13^2} + \frac{1^2}{13^2} = \frac{55}{169}$ or $\boxed{A}$.

14. A point A is selected on circle O with radius 10. Two chords, each with one endpoint at A, are drawn such that they are 45° apart. If one of the chords have length 17, what is the product of the possible values of the length of the other chord?

(A) 85 **(B)** 89 **(C)** 100 **(D)** 170 **(E)** 189

Solution:



We wish to compute $AC \cdot AD$.

We can use Ptolemy's Theorem on quadrilateral $ACBD$. We have $AB \cdot CD = AC \cdot BD + AD \cdot BC$. Because $\angle DAB + \angle BAC = 45^\circ + 45^\circ = 90^\circ$, CD is the diameter of the circle, and thus is equal to 20. This means that $AC \cdot BD + AD \cdot BC = 340$.

Since $\angle DAB = \angle BAC$, $BC = BD$. (This can be easily seen with the Inscribed Angle Theorem.) Since $BC^2 + BD^2 = CD^2 = 20^2$, $BC^2 = BD^2 = 200$, so $BC = BD = 10\sqrt{2}$.

Plugging these back in gives us $AC \cdot 10\sqrt{2} + AD \cdot 10\sqrt{2} = 340$, so $AC + AD = 17\sqrt{2}$. We also know that $AC^2 + AD^2 = CD^2 = 400$. Thus, we can compute $AC \cdot AD = \dfrac{(AC+AD)^2 - (AC^2 + AD^2)}{2} = \dfrac{578 - 400}{2} = 89 \to \boxed{B}$.

Note: An alternate solution is to use trigonometry.

15. What is the maximum value of $\sqrt{1-x^2}(24x+7\sqrt{1-x^2})$ for any real number $-1 \le x \le 1$?

(A) 7 **(B)** 9 **(C)** 16 **(D)** 24 **(E)** 25

Solution: We can replace x with $\cos(\theta)$ to get that the expression we want to maximize is $24\sin(\theta)\cos(\theta)+7\sin^2(\theta)$. This looks super close to the form $(a\sin(\theta)+b\cos(\theta))^2 = a^2\sin^2(\theta)+2ab\sin(\theta)\cos(\theta)+b^2\cos^2(\theta)$. Note that $4^2-3^2=7$ and $2\cdot 3\cdot 4=24$, which leads us to looking at $(4\sin(\theta)+3\cos(\theta))^2 = 16\sin^2(\theta)+24\sin(\theta)\cos(\theta)+9\cos^2(\theta) = 16\sin^2(\theta)+24\sin(\theta)\cos(\theta)+9-9\sin^2(\theta) = 7\sin^2(\theta)+24\sin(\theta)\cos(\theta)+9$. $4\sin(\theta)+3\cos(\theta) = 5\left(\frac{4}{5}\sin(\theta)+\frac{3}{5}\cos(\theta)\right) = 5\left(x+\arcsin(\frac{3}{5})\right)$, which maximizes at 5. Thus, the maximum of $24\sin(\theta)\cos(\theta)+7\sin^2(\theta)$ is $5^2-9=25-9=16 \rightarrow \boxed{C}$.

16. How many ordered triples (a,b,c) of positive integers less than or equal to 20 are there such that $\dfrac{a-b}{c}$, $\dfrac{b-c}{a}$, and $\dfrac{c-a}{b}$ are all integers?

(A) 20 **(B)** 66 **(C)** 101 **(D)** 158 **(E)** 198

Solution: Without loss of generality, suppose a is the largest among the three integers. Then, clearly, a does not divide $|b-c| < b, c$ unless $b-c=0$. Then, that also implies from $\frac{a-b}{c}$ being an integer that $b=c$ divides a. Thus, the possible triples are of the form (kb,b,b) or a permutation where k is a positive integer. Let us ignore overcounts with $k=1$ for now. Then, there are 3 triples for each pair (k,b). To find the number of pairs, do casework on b. We can calculate from 1 to 20 that there are $20+10+6+5+4+3+2+2+2+2+1\cdot 10=66$ pairs (k,b). We then have $3\cdot 66=198$ triples. However, as stated earlier, we overcount when $k=1$, or when the triple is (b,b,b). That happens 20 times, and each time, it is overcounted twice. Thus, our answer is $198-2\cdot 20=158$. $\rightarrow \boxed{D}$.

17. Let ABC be an acute triangle with $\tan(A)=\frac{4}{3}$. Let Ω with center O be the circle passing through A, B, and C. Let X be the foot of the altitude from A to BC. Let H be the intersection of the three altitudes of ABC. Let $D \neq A$ be the intersection of line AO with Ω. Similarly, let $E \neq A$ be the intersection of line AH with Ω. If $AO=5$ and $XB=3XC$, what is the value of DE?

(A) 3 **(B)** $\dfrac{15}{4}$ **(C)** 4 **(D)** $\sqrt{21}$ **(E)** $\dfrac{24}{5}$

Solution: First, note that $DE=\sqrt{AD^2-AE^2}=\sqrt{100-AE^2}$. Thus, it suffices to find AE.

We know that $\angle COB = 2\angle CAB$. Since $\tan(A)=\frac{4}{3}$, $\cos(A)=\frac{3}{5}$, so $\cos(\angle COB)=2\left(\frac{3}{5}\right)^2-1=-\frac{7}{25}$. Therefore, by the Law of Cosines, $CB^2=CO^2+BO^2-2\cdot CO\cdot BO\cdot\frac{-7}{25}$, so $CB^2=5^2+5^2+14=64$. Thus, $BC=8$, so $XB=6$ and $XC=2$.

Now, it is well-known (and easy to prove) that $AH=2R|\cos(A)|=6$. Let $HX=k$. By the Reflection of Orthocenter, $XE=k$ as well.

We can now use power of a point with respect to point X to get $AX\cdot XE=BX\cdot XC$, which gives us $(6+k)\cdot k=12$, or $k^2+6k-12=0$. Thus, $k=\dfrac{-6\pm\sqrt{36+48}}{2}=-3+\sqrt{21}$. Thus, $AE=6+2k=2\sqrt{21}$, which gives us $DE=\sqrt{100-84}=4\rightarrow\boxed{C}$.

Note: Another solution is to use the fact that $BDEC$ is an isosceles trapezoid.

18. Let $r_1, r_2, \cdots, r_{100}$ be the 100 roots (both real and imaginary) of the polynomial

$$\sum_{n=1}^{101} n(-x)^{101-n} = x^{100}-2x^{99}+3x^{98}-4x^{97}+\cdots+99x^2-100x+101$$

with $|r_1|\le|r_2|\le\cdots\le|r_{100}|$. Define a binary operation $a\heartsuit b=a+b-ab$. Let $N=((\cdots(((r_1\heartsuit r_2)\heartsuit r_3)\heartsuit r_4)\cdots)\heartsuit r_{99})\heartsuit r_{100}$. What is the value of $|N|$?

(A) 50 **(B)** 51 **(C)** 52 **(D)** 53 **(E)** 54

Solution: Note that $a\heartsuit b = a+b-ab = 1-(1-a-b+ab) = 1-(1-a)(1-b)$. We can continue in this fashion to get $(a\heartsuit b)\heartsuit c = 1-(1-(1-(1-a)(1-b)))(1-c) = 1-(1-a)(1-b)(1-c)$, and in general, $((\cdots(((a_1\heartsuit a_2)\heartsuit a_3)\heartsuit a_4)\cdots)\heartsuit a_{k-1})\heartsuit a_k = 1-(1-a_1)(1-a_2)\cdots(1-a_k)$. Therefore,
$N = ((\cdots(((r_1\heartsuit r_2)\heartsuit r_3)\heartsuit r_4)\cdots)\heartsuit r_{99})\heartsuit r_{100} = 1-(1-r_1)(1-r_2)\cdots(1-r_{100})$. However, because the polynomial is actually equal to $P(x) = (x-r_1)(x-r_2)\cdots(x-r_{100})$, $N = 1-P(1)$. We can calculate $P(1)$ to be $1-2+3-4+\cdots+99-100+101 = -1-1\cdots-1+101 = -50+101 = 51$. Thus, $N = 1-51 = -50$, so $|N| = 50 \to \boxed{A}$.

Note: An alternate solution is to bash with Vieta's formulas.

19. There are 8 students in a classroom, in which friendship is mutual. Suppose among any three students, there is an odd number of friend pairs. How many possible ways can the students be friends with each other?

(A) 24 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 1024

Solution: Our first claim is that any two people A and B who are connected through a series of friendships are friends. Suppose the people in between are F_1, $F_2, \ldots$, F_k for some nonnegative integer k. Basically, note that the triangle AF_1F_2 has at least two friendships already, namely AF_1 and F_1F_2. This implies that A and F_2 are friends. We then consider AF_2F_3, and get a similar conclusion. By the end, we will have concluded that B and A are friends. This means that the classroom is split into groups, where each person in each group is friends with everyone else in that group. However, we can limit the number of groups to at most 2, because otherwise we could choose three people from three different groups, and there would be 0 friend pairs among them. Thus, we just need to calculate the number of ways to split the 8 students into two groups (could be 1 group, too). Basically, we can either put someone in person 1's group, or in the other group. Thus, the number of ways is 2^7. $\to \boxed{C}$

20. Evan has four positive integers written on the board. He randomly selects three of the positive integers and replaces the integer that was not chosen with the average of the three chosen ones. Evan continues this process until he writes a number that is not an integer for the first time. What is the expected number of integers he would replace given that he starts off with the positive integers $3^{20}, 3^{20}, 3^{21}$ and 3^{21}?

(A) $\frac{83}{3}$ **(B)** $\frac{86}{3}$ **(C)** 41 **(D)** $\frac{143}{3}$ **(E)** $\frac{146}{3}$

Solution: Note that each time we average, we divide by 3. This means we really only have to focus on the power of 3 that is involved. Now, consider how the largest power of 3 that divides all four numbers changes as Evan goes through the process. First, it is clear that no matter what, after the first erasure, the largest power is 3^{19}, and there is only one number on the board divisible by 3^{19} but not 3^{20}. I now claim that we will always be in such a position, but replace 19 with a variable n and 20 with $n+1$. If we are in such a position, note that if we choose the single number divisible by only 3^n, we must end up with a single number divisible by only 3^{n-1}, so we are in a similar position. If we choose the three other numbers, nothing changes (consider what we must have chosen previously to get here). Thus, we basically have a $\frac{3}{4}$ chance of moving n down by 1, and a $\frac{1}{4}$ chance of nothing happening. In the context of the problem, we basically want to go from 19 to -1 because once it is -1, it is not an integer anymore. Note that the expected number of turns it takes to go down 20 is $\frac{20\cdot 4}{3} = \frac{80}{3}$. Now, we add on the extra 1 turn in the beginning and get $\frac{83}{3}$. $\to \boxed{A}$

(Read up on p-adic values and this solution will make a lot more sense).

21. 100 students in the Arvine Unified School District are taking a quiz. Each student randomly submits a real number between 0 and 1. All students submit a different real number. A student's score is the minimum positive difference between his or her number and another student's number. Let M be the maximum number of distinct scores. What

is the probability that there will be M distinct scores?

(A) $\frac{2^{98}}{100!}$ **(B)** $\frac{2^{99}}{100!}$ **(C)** $\frac{2^{98}}{99!}$ **(D)** $\frac{2^{99}}{99!}$ **(E)** $\frac{2^{98}}{98!}$

Solution: Let the numbers chosen be $s_1, s_2, s_3, \cdots, s_{100}$. Let $d_i = s_{i+1} - s_i$ for all $1 \le i \le 99$ (We can assume that all d_i are distinct because the probability two are the same is basically 0; it rarely happens)..

First, I claim that $M = 99$. This is possible if we choose the numbers such that $d_i < d_{i+1}$ for all $1 \le i \le 98$. To see that 100 is not attainable, suppose that everyone had a distinct score. Let d_i be the smallest distance. Then, both s_i's score and s_{i+1}'s score would be $|s_i - s_{i+1}|$, meaning that the scores cannot all be distinct.

Now, in order for $M = 99$ to be achieved, only the pair of chosen numbers with the smallest difference can have the same score. Thus, that means that for every two adjacent numbers chosen, their closest number must be different. This means that both to the left and to the right of the smallest difference pair, the differences must be increasing. In other words, $d_1 > d_2 > \cdots > d_{i-1} > d_i < d_{i+1} < \cdots < d_{98} < d_{99}$. The number of ways this can happen for every i is just selecting $i-1$ of the 98 to be left of it, so there are $\binom{98}{i-1}$ ways. However, $\sum_{i=1}^{99} \binom{98}{i-1} = 2^{98}$ and there are 99! ways to arrange the 99 different numbers, which means our probability is $\frac{2^{98}}{99!} \to \boxed{C}$.

22. Let ABC be a triangle with $AB = 8$, $BC = 7$ and $CA = 5$. Let ω denote the circumcircle of ABC. The tangent to the ω at B meet line AC and the tangent at C to ω at points D and E, respectively. Let the circumcircle of ABE meet segment CD at F. What is the length of EF?

(A) $\frac{35}{8}$ **(B)** $\frac{7\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{645}}{5}$ **(D)** $\frac{5\sqrt{70}}{8}$ **(E)** $\frac{40}{7}$

Solution: First, by law of cosines we have

$$\cos A = \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \frac{1}{2},$$

and so $\angle A = 60°$. Let $\angle A$ denote $\angle BAC$ and similarly for the others. The key idea is that $\triangle FEC \sim \triangle CAB$. Indeed, this is not too hard to see since

$$\angle FEC = \angle FEB - \angle BEC = 180° - \angle A - (180° - 2\angle A) = \angle A,$$

from cyclic quadrilateral $ABEF$. Next, we have $\angle FCE = 180° - \angle ACB - \angle BCE = 180° - \angle C - \angle A = \angle B$. Hence, the claim follows. Now since $\angle A = 60°$, triangle BCE is equilateral, and hence $CE = BC = 7$. Now by similarity,

$$\frac{EF}{5} = \frac{EF}{AC} = \frac{EC}{AB} = \frac{7}{8},$$

which gives a final answer of 35/8, or $\boxed{A}$.

23. Given that $x^4 + ax^3 + bx^2 + 4ax + 16$ has four distinct positive real roots for integers a and b, what is the smallest possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 23 **(E)** 26

Solution: Because the polynomial is almost symmetric, we divide by x^2 to get $x^2 + ax + b + \frac{4a}{x} + \frac{16}{x^2}$. Let $y = x + \frac{4}{x}$. Then, $x^2 + ax + b + \frac{4a}{x} + \frac{16}{x^2} = y^2 - 8 + ay + b = y^2 + ay + b - 8$. Let $c = b - 8$ to simplify calculations, and we wish to minimize $a + c + 8$. Note that the roots of $y^2 + ay + c$ is $\frac{-a \pm \sqrt{a^2-4c}}{2}$. Note that in order for there to be four distinct positive real roots, $y = x + \frac{4}{x} \ge 2\sqrt{4} = 4$. Also, because the roots must be distinct, $y = x + \frac{4}{x} \ne 4$ and $\sqrt{a^2 - 4c} \ne 0$. We know that since $-a \pm \sqrt{a^2 - 4c} > 2 \cdot 4 = 8$ and $\sqrt{a^2 - 4c} \ge 0$, $a < -8$. We can do casework from here.

$a=-9$: $9 \pm \sqrt{81-4c}$. In order to make both values be greater than 8, we want to minimize $81-4c$ while keeping c to still be an integer. The smallest value is obtained when $c=20$, which leads to $9 \pm 1 = 8, 10$. However, this includes an 8, which is not possible. If $c < 20$, then one of the values would be even smaller than 8, so we move on to the next case.

$a=-10$: $10 \pm \sqrt{100-4c}$. We need $\sqrt{100-4c} > 0$ and very small. We can take $c=24$ to be the largest possible value, which gives us $10 \pm 2 = 8, 12$. This once again includes 8, so it is invalid. If $c < 24$, then $\sqrt{100-4c}$ is even larger so there will be a value smaller than 8.

$a=-11$: $11 \pm \sqrt{121-4c}$. We want $\sqrt{121-4c}$ to be small and greater than 0. We can take $c=30$, which leads to $11 \pm 1 = 10, 12$. This works since $10 \neq 12$ and $10, 12 > 8$. We must still check if a smaller value of c satisfies this though. Take $c=29$, and we get $11 \pm \sqrt{5}$. Since $2 < \sqrt{5} < 3$, $11 \pm \sqrt{5} > 8$ and are distinct, so this works as well! In the case where $c \leq 28$, $\sqrt{121-4c} \geq 3$, which means one of the values of y would be less than or equal to 3.

For smaller values of a, note that the sum of $a+c+8$ would actually be larger because the necessary value of c grows at a quadratic rate while a would decrease constantly, so $a+c+8$ is minimized at $a=-11$ and $c=29$ which leads us to our answer of $-11+29+8 = 26 \rightarrow \boxed{E}$.

24. For how many real numbers $1 \leq x \leq 20$ is $\lfloor x \rfloor \{\sin(\pi x^2)\}$ an integer? (Note: $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n and $\{n\} = n - \lfloor n \rfloor$ denotes the fractional part of n.)

(A) 10043 **(B)** 10044 **(C)** 10071 **(D)** 10261 **(E)** 11661

Solution: First, note that $\lfloor x \rfloor \{\sin(\pi x^2)\}$ is an integer if and only if $\lfloor x \rfloor \sin(\pi x^2)$ is an integer. Call any x for which the expression is an integer a solution.

Now, for an integer $1 \leq k \leq 19$, I claim that there are $2k(2k+1)$ solutions for $k \leq x < k+1$. In order for x to be a solution, $\sin(\pi x^2)$ must have a denominator of k (or one that divides k). Note that each of the $2k+1$ intervals $[k^2\pi, (k^2+1)\pi)$, $[(k^2+1)\pi, (k^2+2)\pi)$,..., $[(k^2+2k)\pi, (k+1)^2\pi)$ has $2k$ values for which the sine has a denominator of k (or one that divides k. This is because in each interval, sine goes from 0 to 1 to 0, or 0 to -1 to 0. Basically, we can trace the graph and notice how it will start at 0, go to $\frac{1}{k}$, then $\frac{2}{k}$, ..., to 1, and then back down again to $\frac{1}{k}$ (negate this for the 0 to -1 to 0 case). Thus, our claim is proven.

Now, we need to calculate the sum $1 + \sum_{k=1}^{19} 2k(2k+1)$, where the extra 1 comes from $x = 20$. Note that $\sum_{k=1}^{19} 2k(2k+1) = \sum_{k=1}^{19} 4k^2 + 2k = \frac{4 \cdot 19 \cdot 20 \cdot 39}{6} + \frac{2 \cdot 19 \cdot 20}{2} = 9880 + 380 = 10260$. Adding 1 gives 10261 which is $\boxed{D}$.

25. Define the function $F(a,b)$ for two relatively prime integers $a, b \geq 2$ as the smallest positive multiple of a that leaves a remainder of 1 upon division by b. If a and b are not relatively prime, $F(a,b) = \frac{ab}{2}$. Find $\sum_{a=2}^{15}\sum_{b=2}^{15} F(a,b)$.

(A) 7080.5 **(B)** 7090 **(C)** 7137.5 **(D)** 7200 **(E)** 14400

Solution: First, we prove that when a and b are relatively prime, $F(a,b) + F(b,a) = ab+1$. Note that clearly, $2ab > F(a,b) + F(b,a) \geq a+b > 1$. Next, note that $F(a,b) + F(b,a) \equiv F(b,a) \equiv 1 (\text{mod } a)$, and similarly, $F(a,b) + F(b,a) \equiv 1 (\text{mod } b)$. Thus, $F(a,b) + F(b,a) \equiv 1 (\text{mod } ab)$. But since we bounded $F(a,b) + F(b,a)$ between 1 and $2ab$, the only possible value is then $ab+1$ and we are done.

Now, note that the desired summation is really close to the following sum

$$\sum_{a=2}^{15}\sum_{b=2}^{15} \frac{ab}{2} = \frac{1}{2}(2+3+\ldots+15)(2+3+\ldots+15) = \frac{119^2}{2} = \frac{14161}{2}.$$

The only difference is that we add $\frac{1}{2}$ for each pair of relatively prime a and b. Thus, we need to count the number of pairs of integers from 2 to 15 inclusive that are relatively prime. There are plenty of ways to do this, but we will present the way with the least calculation: PIE on shared prime factors. In total, there are 14^2 pairs. Of those, exactly 7^2 share 2 as a prime factor, 5^2 share 3 as a prime factor, 3^2 share 5 as a prime factor, 2^2 share 7 as a prime factor, 1^2 share 11, 1^2 share 13. Next, we consider when they share two primes, namely 2^2 share 2 and 3, 1^2 share 2 and 5, 1^2 share 3 and 5, and 1^2 share 2 and 7. We do not need to go further because no number from 2 to 15 is divisible by 3 distinct primes. Thus, our the number of relatively prime pairs is $14^2 - 7^2 - 5^2 - 3^2 - 2^2 - 1^2 - 1^2 + 2^2 + 1^2 + 1^2 + 1^2 = 196 - 49 - 25 - 9 + 1 = 114$, so our answer is $\frac{14161}{2} + \frac{114}{2} = \frac{14275}{2} = 7137.5. \rightarrow \boxed{\text{C}}$