OMC 10 Solutions

STEAM for All

January 2021

1 Answer Key

1. C
2. B
3. B
4. C
5. D
6. E
7. D
8. D
9. C
10. D
11. B
12. D
13. B
14. D
15. E
16. D
17. B
18. A
19. D
20. B
21. C
22. C
23. C
24. A
25. E

2 Solutions

1. What is the value of $-7(-5(-4(1-2)-3)-6)-8$?

   (A) -393 **(B)** -85 **(C)** 69 **(D)** 98 **(E)** 406

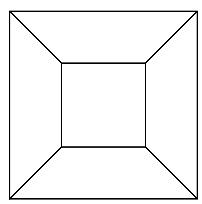
   Solution: $-7(-5(-4(1-2)-3)-6)-8 = -7(-5(4-3)-6)-8 = -7(-11)-8 = 69$, which is $\boxed{C}$.

2. 250 students have registered for SFA's OMC 10/12. A proctor is allowed to watch over at most ten students to ensure no foul play occurs. If 70% of the registrants actually show up to the contest, what is the minimum number of proctors needed?

   (A) 17 **(B)** 18 **(C)** 25 **(D)** 35 **(E)** 36

   Solution: $70\% \cdot 250 = 175$, and the multiple of 10 that is just above that is 180, which means we need 18 proctors, giving $\boxed{B}$.

3. In the figure below, the two squares share a center. If the outer square has side length $\sqrt{5}$ and the inner square has side length 1, what is the area of one of the four congruent trapezoids inside the outer square but outside the inner square?



   (A) $\frac{4}{5}$ **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** $\frac{3}{2}$ **(E)** 2

   Solution: The area of one of the congruent trapezoids must be one-fourth of the area between the two squares. The area between the squares is $\sqrt{5}^2 - 1^2 = 4$, so the area of one of the trapezoids is $\frac{1}{4} \cdot 4 = 1 \rightarrow \boxed{B}$.

4. Janice, Ryan, and Samantha all roll a fair six sided die labeled $1, 2, \cdots, 6$. Given that they all role different numbers, what is the probability Ryan's number will be the largest?

   (A) $\frac{1}{6}$ **(B)** $\frac{55}{216}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

   Solution: Simply $\frac{1}{3}$ because each person has an equal chance to get the largest number, and there are no ties. Thus, our answer is $\boxed{C}$.

5. A sequence is defined by $a_0 = \dfrac{1}{2025}$ and

$$a_{n+1} = a_n + a_n^2 + a_n^3 + \cdots.$$

   What is the value of a_{2021}?

   (A) $\frac{1}{7}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

Solution: Note that $a_{n+1} = \frac{a_n}{1-a_n}$ by the infinite geometric sereis formula. If $a_n = \frac{1}{k}$ for some positive integer k, $a_{n+1} = \frac{a_n}{1-a_n} = \frac{1}{k-1}$. Thus, $a_1 = \frac{1}{2024}$, $a_2 = \frac{1}{2023}$, $\cdots$, $a_{2021} = \frac{1}{4} \to \boxed{D}$.

6. Let $ABCD$ be a trapezoid with $\angle B = 90^\circ$ and $AD = 2 \cdot BC$. What is the sum of all possible angles $\angle BAD$?

(A) 90° **(B)** 120° **(C)** 210° **(D)** 240° **(E)** 270°

Solution: There are two cases. Either AD is parallel to BC or AB is parallel to CD. Consider the former case. In this case we must have $\angle BAD = 90^\circ$. In the latter case, drop the altitude from A onto BC at point X. Then, AXD is a 30-60-90 with the 30 degree angle being at A. There are two possibilities depending on the orientation of triangle ADX. This gives $\angle BAD = 30^\circ$ or $\angle BAD = 90^\circ + 60^\circ = 150^\circ$. Adding up all the values gives $\boxed{270^\circ}$ or $\boxed{E}$ as the final answer.

7. Ben chooses three vertices of a regular hexagon at random and draws the triangle with vertices at these three points. What is the probability that the area of this triangle is at least one-third of the area of the hexagon?

(A) $\dfrac{1}{10}$ **(B)** $\dfrac{3}{10}$ **(C)** $\dfrac{2}{5}$ **(D)** $\dfrac{7}{10}$ **(E)** $\dfrac{9}{10}$

Solution: Without loss of generality, assume the side length of the hexagon is 1. Then, the area is equal to $\dfrac{3\sqrt{3}}{2}$, so we want all cases where the area of the triangle is greater than or equal to $\dfrac{\sqrt{3}}{2}$. There are three noncongruent triangles that Ben may have formed.

Case 1 (Three consecutive vertices): Here, there are 6 different ways to select these vertices (just choose which vertex of the six will be in the middle). In this case, the triangle is a 30-30-120 triangle, and thus is equal to two 30-60-90 triangles put together. Therefore, the area of one of these triangles is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{2}$

Case 2 (Two adjacent vertices and one vertex not adjacent to either of the other two): Here, there are 6 ways to select the two adjacent vertices and 2 ways to select the other. This is a right triangle, so its area is equal to $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2} \geq \frac{\sqrt{3}}{2}$.

Case 3 (Every other vertex): There are only 2 cases here. The area of the equilateral triangle is just $\frac{\sqrt{3}}{4}(\sqrt{3})^2 = \frac{3\sqrt{3}}{4} \geq \frac{\sqrt{3}}{2}$ because the side length of the triangle is $\sqrt{3}$.

Thus, in cases 2 and 3 our property is satisfied so our probability is $\dfrac{12+2}{20} = \dfrac{7}{10} \to \boxed{D}$.

8. A positive integer has 16 factors, k of which are even. What is the sum of all possible values of k?

(A) 15 **(B)** 31 **(C)** 34 **(D)** 49 **(E)** 65

Solution: Note that the basic condition is that the prime factorization has 2^n where $n+1$ divides 16. Among the divisors, we must have exactly $\frac{n}{n+1}$ even, because that is how many have a nonzero power of 2. Since $n+1$ can be $1, 2, 4, 8, 16$, we know $16 \cdot \frac{n}{n+1}$ can be $0, 8, 12, 14, 15$, summing to 49. $\to \boxed{D}$

9. John has a deck of 2020 cards numbered 1 through 2020 in that order, with the bottom card labeled 1 and the top card labeled 2020. At each step, he removes the top and bottom cards from the deck and places both cards, in that order, at the top of the remaining stack. For example, if the top and bottom cards in the deck were labeled A and B, respectively, then after one step, the top two cards in the deck would be A and B, in that order. After 2100 steps, what is the number on the second card?

(A) 79 **(B)** 80 **(C)** 81 **(D)** 1939 **(E)** 1940

Solution: Note that the first card is always the same, and that we basically cycle through the rest of the cards. In other words, the 2nd card is initially 2019, and after moves we get 1, 2, 3,$\ldots$, and we get back to 2019 with 2019 moves. We are left with 81 moves, or 80 moves after 1. That gives us $1 + 80 = 81$, which is $\boxed{C}$.

10. Let a_n be a strictly increasing sequence of positive integers defined for all integers $n \geq 1$ such that the sum of any 2020 consecutive terms is divisible by 2020. If $a_i = i$ for all integers $1 \leq i \leq 2019$, what is the smallest possible value of a_{4040}?

(A) 4040 **(B)** 5050 **(C)** 6060 **(D)** 7070 **(E)** 8080

Solution: First, we claim that 2020 must divide $a_{i+2020} - a_i$ for all positive integers i. This is because both $a_i + a_{i+1} + \ldots + a_{i+2019}$ and $a_i + a_{i+1} + \ldots + a_{i+2019}$ are both divisble by 2020, and taking the difference gives $a_{i+2020} - a_i$. Next, note that since $a_1 + a_2 + \ldots + a_{2019} = \frac{2019 \cdot 2020}{2} = 2019 \cdot 1010$, a_{2020} is at least 3030. In order for a_{4040} to be as small as possible, we must choose $a_{2020} = 3030$. Thus, the smallest values for $a_{2021}, \ldots, a_{4039}$ are $4041, \ldots 6059$, respectively. (namely 4040 plus a_1 through a_{2019}). It then follows that since $a_{2020} + 2020 < 6059$, we must have $a_{4040} \geq a_{2020} + 4040 = 7070$, and that is our answer $\boxed{D}$.

11. Olivia writes down the binary representation for all positive integers $1 \leq n \leq 1024$. How many more 1s does Olivia write than 0s?

(A) 1013 **(B)** 1014 **(C)** 1015 **(D)** 1023 **(E)** 1024

Solution: Consider the numbers from 2^k to $2^{k+1} - 1$ for $0 \leq k \leq 9$. Note that they are the $k + 1$ digit binary numbers, and that if we remove the first digit of 1, we get a string of k digits that can be any combination of 0 and 1. In particular, if we look at the numbers in the interval as a whole, the 0's and 1's balance out except for the leading digit that we removed. Thus, we get 2^k more $1's$ than $0's$. Summing from $k = 0$ to $k = 9$ gives $1 + 2 + \ldots + 2^9 = 1023$. Now, consider $1024 = 10000000000_2$. It has 9 more 0's than 1's, so we subtract 9 from 1023 to get 1014, or $\boxed{B}$.

12. Let $ABCD$ be a rectangle with $AB = 13$ and $BC = 5$. Curtis folds rectangle $ABCD$ along a line ℓ passing through A such that point B lies on segment CD. He notices that ℓ intersects segment BC at a point X. What is the length of BX?

(A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{13}{5}$ **(E)** 3

Solution: Let B fold to Y. Then, $AY = AB = 13$. Thus, $\triangle ADY$ is a 5-12-13 triangle. Specifically, $DY = 12$ and $YC = 1$. Now, our claim is that $\triangle ABX \sim \triangle BCY$. First, note that $BY \perp AX$, so $\angle YBC = 90° - \angle ABY = \angle XAB$. Since $\triangle ABX$ and $\triangle BCY$ are both right triangles, it follows that they are similar. Thus, $BX = AB \cdot \frac{CY}{BC} = \frac{13 \cdot 1}{5} = \frac{13}{5}$ and the answer is $\boxed{D}$.

13. Let $Q(x)$ be a quadratic with leading coefficient one, real coefficients, two real roots, sum of roots s, and product of roots p. If $s = p$, how many of the following **must** be true?

I: The linear and constant coefficient are equal.

II: For every positive real number a, there exists a polynomial $Q(x)$ that satisfies the given conditions and has $p = a$.

III: For every positive real number d, there exists exactly two polynomials $Q(x)$ that satisfies the given conditions and has the positive difference of the two roots equal to d.

IV: For every positive integer k, there exists a polynomial $Q(x)$ that satisfies the given conditions and has at least one root equal to k.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: From the conditions, we can figure out that $Q(x) = x^2 - sx + s$. Because both roots are real, we must also have $s^2 - 4s$, the discriminant, greater than 0, so $s > 4$ or $s < 0$. Let's get rid of the obvious ones: I and II. Next, we note that the for III, the positive difference is exactly the square root of the discriminant of $s^2 - 4s$. It is clear that $s^2 - 4s$ can be any positive real number, so there always exists a s for which $d = \sqrt{s^2 - 4s}$ and III is true. As for IV, note that $Q(1) = 1$ no matter what, so $k = 1$ doesn't work. Thus, only III must be true, and our answer is $\boxed{B}$.

14. Three spheres of radius 1 are all externally tangent to each other and externally tangent to a plane P. There exists an unique sphere S such that S is tangent to P and all three spheres of radius 1. Find the radius of S.

(A) $\frac{\sqrt{3}}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{\sqrt{3}}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

Solution: Let O_1, O_2, O_3 be the centers of the larger spheres and let X be the center of the smaller sphere that is tangent to the plane and the three spheres and let r be the radius of this sphere. Connect all four centers with each other. Let Y be the foot of point X onto triangle $O_1O_2O_3$. Note that O_1XY is a right triangle with hypotenuse O_1X. We know that $O_1X = r + 1$ and $XY = 1 - r$. To find O_1Y, we note that $O_1O_2O_3$ is an equilateral triangle with center Y, so O_1Y is $\frac{\sqrt{3}}{3} \cdot O_1O_2 = \frac{2\sqrt{3}}{3}$ (by 30-60-90 triangle properties). Thus,

$$(1 - r)^2 + \frac{4}{3} = (1 + r)^2$$

$$\frac{4}{3} = 4r$$

This means $r = \frac{1}{3} \to \boxed{D}$.

15. Let $f(n)$ be the sum of all positive divisors of n. The smallest interval containing all possible values of $\dfrac{f(12n)}{f(n)}$ can be expressed as $(a, b]$. What is the value of $b - a$?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: We know that for any $n = p_1^{e_1}p_2^{e_2}p_k^{e_k}$, $f(n) = (1+p_1+\cdots+p_1^{e_1})(1+p_2+\cdots+p_2^{e_2})\cdots(1+p_k+\cdots+p_k^{e_k})$. (If you have never seen this before, try proving it!) Also, note that any prime factors of n other than 2 or 3 will cancel out in $\frac{f(12n)}{f(n)}$, so we only need to care about the first two primes. Therefore,

$$\frac{f(12n)}{f(n)} = \frac{(1 + 2^1 + 2^2 + \cdots + 2^{e_1+2})(1 + 3^1 + 3^2 + \cdots + 3^{e_2+1})}{(1 + 2^1 + 2^2 + \cdots + 2^{e_1})(1 + 3^1 + 3^2 + \cdots + 3^{e_2})}.$$

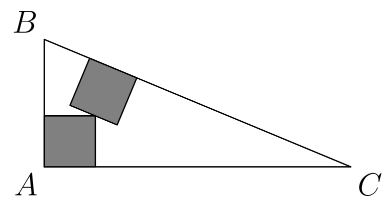
This value is minimized when e_1 and e_2 are extremely large, in which case the fraction is $4 \cdot 3 = 12$. The fraction is maximized when $e_1 = e_2 = 0$, in which case it is equal to $7 \cdot 4 = 28$. Thus, our answer is $28 - 12 = 16 \to \boxed{E}$.

16. Define $P(x)$ to be a degree 40 polynomial with real coefficients such that for all real x, $xP(x+40) = (x+40)P(x+39)$. If $P(-1) = 1$, find $P(42)$.

(A) -1772 **(B)** -861 **(C)** 42 **(D)** 861 **(E)** 1722

Solution: Our claim is that $P(x) = Cx(x-1)(x-2)\dots(x-39)$ for some constant C. First, let us change the condition of the problem by moving x up by 39. That gives $(x-39)P(x+1) = (x+1)P(x)$. Note that 39 is a root of $P(x)$, and so $P(39) = 0$. Now, plug in $x = 38$ and we get $0 = -P(39) = 39P(38)$, so 38 is a root as well. We can repeat this process down and prove that $x = 0$ is a root, and that is the farthest we can go. But note that we have 40 distinct roots, and $P(x)$ is degree 40, so we know $P(x) = Cx(x-1)(x-2)\dots(x-39)$. Now, note that $P(-1) = C\cdot(-1\cdot -2\cdot -3\cdot \dots \cdot -40) = 40!C$. Meanwhile, $P(42) = C\cdot(42\cdot 41\cdot 40\dots\cdot 3) = C\frac{42!}{2}$, so $P(42) = \frac{42!C}{2} = \frac{42!C}{2\cdot P(-1)} = \frac{42!C}{2\cdot 40!C} = 21\cdot 41 = 861$, which is $\boxed{D}$.

17. In right triangle ABC, two identical squares are placed as shown below. If the perimeter of one of the squares is exactly one-tenth of the perimeter of ABC, what is the ratio of the shaded area to the area of ABC?



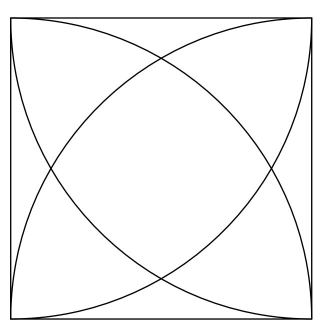
(A) $\frac{1}{20}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{5}$

Solution: We claim that the sidelength of the square is equal to the inradius r of the triangle. This is clear since the top right corner of the bottom square is equidistant from all three sides, with the distance being equal to the inradius. Let s be the semiperimeter of the triangle. Then, we know $4r = \frac{1}{10}2s$, or $r = \frac{s}{20}$. Note that the area of the shaded area is $2r^2$ while the area of the triangle is rs, so $\frac{2r^2}{rs} = \frac{2r}{s} = \frac{1}{10}$, which is $\boxed{B}$.

18. In square $ABCD$ of side length 1, a point P is selected inside the square. Let R be the region that the interior of $ABCD$ sweeps out when rotating about the point P. Let S be the set of points P such that the area of R is π. What is the area enclosed by S?

(A) $\frac{\pi}{3} - \sqrt{3} + 1$ **(B)** $\frac{\pi}{2} - \frac{\pi\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + 1$ **(C)** $\frac{\pi}{8}$ **(D)** $\pi - \frac{\pi\sqrt{3}}{2}$ **(E)** $\frac{\sqrt{3}}{4}$

Solution: The region swept out by the square is a circle centered at P with radius $\max(PA, PB, PC, PD)$. Hence, we simply require that $\max(PA, PB, PC, PD) = 1$. To find the area of this, draw a circle of radius 1 around each of A, B, C and D. The set of points lying inside of all these circles is the region bounded by S.



Denote the area of the pillow center as C. Denote the "corner" area by A and the "lateral" area by B. Then, we get the following system of equations:

$$B + 2A + C = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$
$$4B + 4A + C = 1$$
$$A + 2B = 1 - \frac{\pi}{4}.$$

Solving gives $C = \frac{\pi}{3} - \sqrt{3} + 1$, or answer choice $\boxed{A}$.

19. How many ordered triples (a, b, c) of positive integers less than or equal to 20 are there such that $\dfrac{a-b}{c}$, $\dfrac{b-c}{a}$, and $\dfrac{c-a}{b}$ are all integers?

(A) 20 **(B)** 66 **(C)** 101 **(D)** 158 **(E)** 198

Solution: Without loss of generality, suppose a is the largest among the three integers. Then, clearly, a does not divide $|b-c| < b, c$ unless $b - c = 0$. Then, that also implies from $\frac{a-b}{c}$ being an integer that $b = c$ divides a. Thus, the possible triples are of the form (kb, b, b) or a permutation where k is a positive integer. Let us ignore overcounts with $k = 1$ for now. Then, there are 3 triples for each pair (k, b). To find the number of pairs, do casework on b. We can calculate from 1 to 20 that there are $20+10+6+5+4+3+2+2+2+2+1\cdot 10 = 66$ pairs (k, b). We then have $3 \cdot 66 = 198$ triples. However, as stated earlier, we overcount when $k = 1$, or when the triple is (b, b, b). That happens 20 times, and each time, it is overcounted twice. Thus, our answer is $198 - 2\cdot 20 = 158$. $\rightarrow \boxed{D}$.

20. Let ABC be a right triangle with at least 2 integer sides. Let A', B', and C' be the reflections of A across BC, B across CA, and C across AB respectively. How many noncongruent triangles ABC are there such that triangle $A'B'C'$ has area 2025?

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 32

Solution: Without loss of generality, assume the right angle is at C. First, note that $ABA'B'$ is a parallelogram, so $AB||AB'$. Next, note that $CC' \perp AB$, so $CC' \perp A'B'$. Suppose CC' meets $A'B'$ at X. Then, note that CX is the distance from C to $B'A'$, which is equal to the distance from C to AB, which is equal to the distance from C' to AB. Thus, we know $C'X = 3CX$, and so the area of $\triangle A'B'C'$ is 3 times the area of $\triangle A'B'C$. In other words, $AC \cdot BC = A'C \cdot B'C = 2\frac{2025}{3} = 1350$. In effect, we want to find the number of unordered pair of integers that multiply to 1350. This is just half the number of divisors (noting that 1350 is not a perfect square). $1350 = 2 \cdot 3^3 \cdot 5^2$, so it has $(1+1)(3+1)(2+1) = 24$ divisors. The answer is then $\frac{24}{2} = 12$, or $\boxed{B}$.

21. Define the function $f\left(\frac{m}{n}\right)$ to be the numerator of the fraction $\frac{m}{n}$ when put in simplest form. For example, $f\left(\frac{6}{8}\right) = 3$ and $f\left(\frac{2}{5}\right) = 2$. What are the last two digits of the sum

$$f\left(\frac{1}{2^{2020}}\right) + f\left(\frac{2}{2^{2020}}\right) + f\left(\frac{3}{2^{2020}}\right) + \cdots + f\left(\frac{2^{2020}-1}{2^{2020}}\right)?$$

(A) 01 **(B)** 05 **(C)** 25 **(D)** 27 **(E)** 75

Solution: Note that the fractions with denominator 2^{2020} once simplified have an odd numerator. Namely, they are $1, 3, \ldots, 2^{2020} - 1$. Similarly, the fractions with denominator 2^{2019} once simplified also have an odd numerator, but they are $1, 3, \ldots, 2^{2019} - 1$. We can see a clear pattern from here. Thus, our answer is simply $(1+3+\ldots+2^{2020}-1)+(1+3+\ldots+2^{2019}-1)+\ldots+(2^1-1) = 2^{4038}+2^{4036}+\ldots+1 = \frac{4^{2020}-1}{4-1}$. To find the last two digits, simply consider the answer modulo 4 and 25. Note that $\phi(25) = 20$ where ϕ is Euler's Totient function, so 25 divides $4^{2020} - 1$. Next, note that $2^{4038} + 2^{4036} + \ldots + 1 \equiv 1 (\text{mod } 4)$. From this info, we can tell that the last two digits are 25, which gives $\boxed{C}$.

22. There are 8 students in a classroom, in which friendship is mutual. Suppose among any three students, there is an odd number of friend pairs. How many possible ways can the students be friends with each other?

(A) 24 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 1024

Solution: Our first claim is that any two people A and B who are connected through a series of friendships are friends. Suppose the people in between are F_1, F_2,..., F_k for some nonnegative integer k. Basically, note that the triangle AF_1F_2 has at least two friendships already, namely AF_1 and F_1F_2. This implies that A and F_2 are friends. We then consider AF_2F_3, and get a similar conclusion. By the end, we will have concluded that B and A are friends. This means that the classroom is split into groups, where each person in each group is friends with everyone else in that group. However, we can limit the number of groups to at most 2, because otherwise we could choose three people from three different groups, and there would be 0 friend pairs among them. Thus, we just need to calculate the number of ways to split the 8 students into two groups (could be 1 group, too). Basically, we can either put someone in person 1's group, or in the other group. Thus, the number of ways is 2^7. $\rightarrow \boxed{C}$

23. 100 students in the Arvine Unified School District are taking a quiz. Each student randomly submits a real number between 0 and 1. All students submit a different real number. A student's score is the minimum positive difference between his or her number and another student's number. Let M be the maximum number of distinct scores. What is the probability that there will be M distinct scores?

(A) $\frac{2^{98}}{100!}$ **(B)** $\frac{2^{99}}{100!}$ **(C)** $\frac{2^{98}}{99!}$ **(D)** $\frac{2^{99}}{99!}$ **(E)** $\frac{2^{98}}{98!}$

Solution: Let the numbers chosen be $s_1, s_2, s_3, \cdots, s_{100}$. Let $d_i = s_{i+1} - s_i$ for all $1 \le i \le 99$ (We can assume that all d_i are distinct because the probability two are the same is basically 0; it rarely happens).

First, I claim that $M = 99$. This is possible if we choose the numbers such that $d_i < d_{i+1}$ for all $1 \le i \le 98$. To see that 100 is not attainable, suppose that everyone had a distinct score. Let d_i be the smallest distance. Then, both s_i's score and s_{i+1}'s score would be $|s_i - s_{i+1}|$, meaning that the scores cannot all be distinct.

Now, in order for $M = 99$ to be achieved, only the pair of chosen numbers with the smallest difference can have the same score. Thus, that means that for every two adjacent numbers chosen, their closest number must be different. This means that both to the left and to the right of the smallest difference pair, the differences must be increasing. In other words, $d_1 > d_2 > \cdots > d_{i-1} > d_i < d_{i+1} < \cdots < d_{98} < d_{99}$. The number of ways this can happen for every i is just selecting $i-1$ of the 98 to be left of it, so there are $\binom{98}{i-1}$ ways. However, $\sum_{i=1}^{99} \binom{98}{i-1} = 2^{98}$ and there are 99! ways to arrange the 99 different numbers, which means our probability is $\frac{2^{98}}{99!} \rightarrow \boxed{C}$.

24. Let ABC be a triangle with $AB = 8$, $BC = 7$ and $CA = 5$. Let ω denote the circumcircle of ABC. The tangent to the ω at B meet line AC and the tangent at C to ω at points D and E, respectively. Let the circumcircle of ABE meet segment CD at F. What is the length of EF?

(A) $\frac{35}{8}$ **(B)** $\frac{7\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{645}}{5}$ **(D)** $\frac{5\sqrt{70}}{8}$ **(E)** $\frac{40}{7}$

Solution: First, note that $\angle A = 60°$. Let $\angle A$ denote $\angle BAC$ and similarly for the others. The key idea is that $\triangle FEC \sim \triangle CAB$. Indeed, this is not too hard to see since

$$\angle FEC = \angle FEB - \angle BEC = 180° - \angle A - (180° - 2\angle A) = \angle A,$$

from cyclic quadrilateral $ABEF$. Next, we have $\angle FCE = 180° - \angle ACB - \angle BCE = 180° - \angle C - \angle A = \angle B$. Hence, the claim follows. Now since $\angle A = 60°$, triangle BCE is equilateral, and hence $CE = BC = 7$. Now by similarity,

$$\frac{EF}{5} = \frac{EF}{AC} = \frac{EC}{AB} = \frac{7}{8},$$

which gives a final answer of 35/8, or $\boxed{A}$.

25. Given that $x^4 + ax^3 + bx^2 + 4ax + 16$ has four distinct positive real roots for integers a and b, what is the smallest possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 23 **(E)** 26

Solution: Because the polynomial is almost symmetric, we divide by x^2 to get $x^2 + ax + b + \frac{4a}{x} + \frac{16}{x^2}$. Let $y = x + \frac{4}{x}$. Then, $x^2 + ax + b + \frac{4a}{x} + \frac{16}{x^2} = y^2 - 8 + ay + b = y^2 + ay + b - 8$. Let $c = b - 8$ to simplify calculations, and we wish to minimize $a + c + 8$. Note that the roots of $y^2 + ay + c$ is $\frac{-a \pm \sqrt{a^2-4c}}{2}$. Note that in order for there to be four distinct positive real roots, $y = x + \frac{4}{x} \geq 2\sqrt{4} = 4$. Also, because the roots must be distinct, $y = x + \frac{4}{x} \neq 4$ and $\sqrt{a^2 - 4c} \neq 0$. We know that since $-a \pm \sqrt{a^2 - 4c} > 2 \cdot 4 = 8$ and $\sqrt{a^2 - 4c} \geq 0$, $a < -8$. We can do casework from here.

$a = -9$: $9 \pm \sqrt{81 - 4c}$. In order to make both values be greater than 8, we want to minimize $81 - 4c$ while keeping c to still be an integer. The smallest value is obtained when $c = 20$, which leads to $9 \pm 1 = 8, 10$. However, this includes an 8, which is not possible. If $c < 20$, then one of the values would be even smaller than 8, so we move on to the next case.

$a = -10$: $10 \pm \sqrt{100 - 4c}$. We need $\sqrt{100 - 4c} > 0$ and very small. We can take $c = 24$ to be the largest possible value, which gives us $10 \pm 2 = 8, 12$. This once again includes 8, so it is invalid. If $c < 24$, then $\sqrt{100 - 4c}$ is even larger so there will be a value smaller than 8.

$a = -11$: $11 \pm \sqrt{121 - 4c}$. We want $\sqrt{121 - 4c}$ to be small and greater than 0. We can take $c = 30$, which leads to $11 \pm 1 = 10, 12$. This works since $10 \neq 12$ and $10, 12 > 8$. We must still check if a smaller value of c satisfies this though. Take $c = 29$, and we get $11 \pm \sqrt{5}$. Since $2 < \sqrt{5} < 3$, $11 \pm \sqrt{5} > 8$ and are distinct, so this works as well! In the case where $c \leq 28$, $\sqrt{121 - 4c} \geq 3$, which means one of the values of y would be less than or equal to 3.

For smaller values of a, note that the sum of $a + c + 8$ would actually be larger because the necessary value of c grows at a quadratic rate while a would decrease constantly, so $a + c + 8$ is minimized at $a = -11$ and $c = 29$ which leads us to our answer of $-11 + 29 + 8 = 26 \rightarrow \boxed{E}$.