Team Round

October 24th, 2020

1. **[2]** Let $f(n) = n^2 + 6n + 11$ be a function defined on positive integers. Find the sum of the first three prime values $f(n)$ takes on.

2. **[2]** Let A be a set of 2020 distinct real numbers. Call a number *scarily epic* if it can be expressed as the product of two (not necessarily distinct) numbers from A. What is the minimum possible number of distinct *scarily epic* numbers?

3. **[3]** Let ABC be a triangle with $AB = 30$, $BC = 14$, and $CA = 26$. Let N be the center of the equilateral triangle constructed externally on side AB. Let M be the center of the square constructed externally on side BC. Given that the area of quadrilateral $ACMN$ can be expressed as $a + b\sqrt{c}$ for positive integers a, b and c such that c is not divisible by the square of any prime, compute $a + b + c$.

4. **[3]** There are 5 tables in a classroom. Each table has 4 chairs with a child sitting on it. All the children get up and randomly sit in a seat. Two people that sat at the same table before are not allowed to sit at the same table again. Assuming tables and chairs are distinguishable, if the number of different classroom arrangements can be written as $2^a 3^b 5^c$, what is $a + b + c$?

5. **[5]** In acute triangle ABC, the lines tangent to the circumcircle of ABC at A and B intersect at point D. Let E and F be points on CA and CB such that $DECF$ forms a parallelogram. Given that $AB = 20$, $CA = 25$ and $\tan C = 4\sqrt{21}/17$, the value of EF may be expressed as m/n for relatively prime positive integers m and n. Compute $m + n$.

6. **[5]** A triple of integers (a, b, c) is said to be γ-*special* if $a \le \gamma(b + c)$, $b \le \gamma(c + a)$ and $c \le \gamma(a + b)$. For each integer triple (a, b, c) such that $1 \le a, b, c \le 20$, Kodvick writes down the smallest value of γ such that (a, b, c) is γ-special. How many distinct values does he write down?

7. **[6]** In triangle ABC, let N and M be the midpoints of AB and AC, respectively. Point P is chosen on the arc BC not containing A of the circumcircle of ABC such that $BNMP$ is cyclic. Given $BC = 28$, $AC = 30$ and $AB = 26$, the value of AP may be expressed as $m/\sqrt{n}$ for positive integers m and n, where n is not divisible by the square of any prime. Compute $m + n$.

8. **[7]** You have been kidnapped by a witch and are stuck in the *Terrifying Tower*, which has an infinite number of floors, starting with floor 1, each initially having 0 boxes. The witch allows you to do the following two things:

   - For a floor i, put 2 boxes on floor $i + 5$, 6 on floor $i + 4$, 13 on floor $i + 3$, 12 on floor $i + 2$, 8 on floor $i + 1$, and 1 on floor i, or remove the corresponding number of boxes from each floor if possible.
   - For a floor i, put 1 box on floor $i + 4$, put 3 boxes on floor $i + 3$, 6 on floor $i + 2$, 5 on floor $i + 1$, and 3 on floor i, or remove the corresponding number of boxes from each floor if possible.

   At the end, suppose the witch wishes to have exactly n boxes in the tower. Specifically, she wants them to be on the first 10 floors. Let $T(n)$ be the number of distinct distributions of these n boxes that you can make. Find $\sum_{n=1}^{15} T(n)$.

9. **[8]** A sequence of nonzero complex numbers $a_1, a_2, \ldots, a_{2020}$ satisfies $a_3 = a_2^2 + 2a_1a_2$ and

$$\frac{a_{n+2}}{a_{n+1}} - \frac{a_{n+1}}{a_n} = a_n + a_{n+1},$$

for all $2018 \geq n \geq 2$. Given $a_2 - a_{2020} = 2025$, how many integers $0 \leq a_1 \leq 2020$ are there, such that $a_1 + a_2 + \cdots + a_{2019}$ is a real number?

10. **[9]** Let $p = 47$ be a prime. Call a function f defined on the integers *lit* if $f(x)$ is an integer from 1 to p inclusive and $f(x+p) = f(x)$ for all integers x. How many *lit* functions g are there such that for all integers x, p divides $g(x^2) - g(x) - x^8 + x$?