OAA OMC
Orange Math Competitions

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OMC 12

Orange Mathematics Competitions
Saturday, January 23, 2021

INSTRUCTIONS

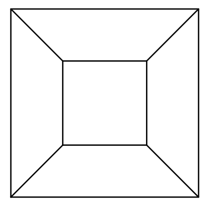
1. DO NOT LOOK AT THE PROBLEMS UNTIL YOU ARE READY TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 12 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/` and mark you answer to each problem on the AMC 12 Answer Sheet with a number 2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the OMC, **you must submit your answers using the Submission Form found at** `https://tinyurl.com/omc12submission`**. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only pencils, erasers, rulers, and scratch paper are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10/12 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/`. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, sign your name in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-1012/`.
9. Enjoy the problems!

The Committee on the Orange Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

1. If 15% of x is equal to 60% of y, what is the value of $\frac{x}{y}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

2. In the figure below, the two squares share a center. If the outer square has side length $\sqrt{5}$ and the inner square has side length 1, what is the area of one of the four congruent trapezoids inside the outer square but outside the inner square?



(A) $\frac{4}{5}$ **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** $\frac{3}{2}$ **(E)** 2

3. Madison writes down the numbers 24 and 36 on a whiteboard. Every second, she replaces the two numbers on the board with their greatest common divisor and their least common multiple. After 2021 seconds, what is the sum of the two numbers on the whiteboard?

(A) 60 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

4. There are N people in a circular pool of diameter 12 feet. To follow social distancing rules, no two people are less than 6 feet from each other. What is the maximum possible value of N?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

5. Ben chooses three vertices of a regular hexagon at random and draws the triangle with vertices at these three points. What is the probability that the area of this triangle is at least one-third of the area of the hexagon?

(A) $\frac{1}{10}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{7}{10}$ **(E)** $\frac{9}{10}$

6. Define $p(r)$ to be the number of lattice points in the region enclosed by $x^2 + y^2 \leq r$. For which positive integers r is $p(r)$ odd?

(A) Odd perfect squares **(B)** Even perfect squares
(C) All perfect squares **(D)** Non-squares **(E)** All positive integers

7. How many real solutions are there to the equation

$$\log_2(\log_4(x^2)) = \log_4(\log_2(x^2))?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

8. A positive integer has 16 factors, k of which are even. What is the sum of all possible values of k?

(A) 15 **(B)** 31 **(C)** 34 **(D)** 49 **(E)** 65

9. Let $f(n)$ denote the product of all the divisors of n. For how many positive integers $2 \leq n \leq 1000$ is $\log_n(f(n))$ an integer?

(A) 30 **(B)** 31 **(C)** 968 **(D)** 969 **(E)** 970

10. Let $ABCD$ be a rectangle with $AB = 13$ and $BC = 5$. Curtis folds rectangle $ABCD$ along a line ℓ passing through A such that point B lies on segment CD. He notices that ℓ intersects segment BC at a point X. What is the length of BX?

(A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{13}{5}$ **(E)** 3

11. Let $Q(x)$ be a quadratic with leading coefficient one, real coefficients, two real roots, sum of roots s, and product of roots p. If $s = p$, how many of the following **must** be true?

    I: The linear and constant coefficient are equal.

    II: For every positive real number a, there exists a polynomial $Q(x)$ that satisfies the given conditions and has $p = a$.

    III: For every positive real number d, there exists exactly two polynomials $Q(x)$ that satisfies the given conditions and has the positive difference of the two roots equal to d.

    IV: For every positive integer k, there exists a polynomial $Q(x)$ that satisfies the given conditions and has at least one root equal to k.

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

12. Three spheres of radius 1 are all externally tangent to each other and externally tangent to a plane P. There exists an unique sphere S such that S is tangent to P and all three spheres of radius 1. Find the radius of S.

    (A) $\frac{\sqrt{3}}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{\sqrt{3}}{6}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

13. Let x_1 and x_2 be two real numbers, each randomly selected in the interval $(0, 1)$. What is the probability that $\lfloor \log_3(x_1) \rfloor + \lfloor \log_3(x_2) \rfloor$ is a multiple of 3?

    (A) $\frac{55}{169}$ **(B)** $\frac{57}{169}$ **(C)** $\frac{5}{13}$ **(D)** $\frac{7}{13}$ **(E)** $\frac{55}{144}$

14. A point A is selected on circle O with radius 10. Two chords, each with one endpoint at A, are drawn such that they are $45°$ apart. If one of the chords have length 17, what is the product of the possible values of the length of the other chord?

    (A) 85 **(B)** 89 **(C)** 100 **(D)** 170 **(E)** 189

15. What is the maximum value of $\sqrt{1-x^2}(24x + 7\sqrt{1-x^2})$ for any real number $-1 \leq x \leq 1$?

    (A) 7 **(B)** 9 **(C)** 16 **(D)** 24 **(E)** 25

16. How many ordered triples (a, b, c) of positive integers less than or equal to 20 are there such that $\frac{a-b}{c}$, $\frac{b-c}{a}$, and $\frac{c-a}{b}$ are all integers?

    (A) 20 **(B)** 66 **(C)** 101 **(D)** 158 **(E)** 198

17. Let ABC be an acute triangle with $\tan(A) = \frac{4}{3}$. Let Ω with center O be the circle passing through A, B, and C. Let X be the foot of the altitude from A to BC. Let H be the intersection of the three altitudes of ABC. Let $D \neq A$ be the intersection of line AO with Ω. Similarly, let $E \neq A$ be the intersection of line AH with Ω. If $AO = 5$ and $XB = 3XC$, what is the value of DE?

    (A) 3 **(B)** $\frac{15}{4}$ **(C)** 4 **(D)** $\sqrt{21}$ **(E)** $\frac{24}{5}$

18. Let $r_1, r_2, \cdots, r_{100}$ be the 100 roots (both real and imaginary) of the polynomial

    $$\sum_{n=1}^{101} n(-x)^{101-n} = x^{100} - 2x^{99} + 3x^{98} - 4x^{97} + \cdots + 99x^2 - 100x + 101$$

    with $|r_1| \leq |r_2| \leq \cdots \leq |r_{100}|$. Define a binary operation $a \heartsuit b = a + b - ab$. Let $N = ((\cdots(((r_1 \heartsuit r_2) \heartsuit r_3) \heartsuit r_4) \cdots) \heartsuit r_{99}) \heartsuit r_{100}$. What is the value of $|N|$?

    (A) 50 **(B)** 51 **(C)** 52 **(D)** 53 **(E)** 54

19. There are 8 students in a classroom, in which friendship is mutual. Suppose among any three students, there is an odd number of friend pairs. How many possible ways can the students be friends with each other?

(A) 24 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 1024

20. Evan has four positive integers written on the board. He randomly selects three of the positive integers and replaces the integer that was not chosen with the average of the three chosen ones. Evan continues this process until he writes a number that is not an integer for the first time. What is the expected number of integers he would replace given that he starts off with the positive integers $3^{20}, 3^{20}, 3^{21}$ and 3^{21}?

(A) $\frac{83}{3}$ **(B)** $\frac{86}{3}$ **(C)** 41 **(D)** $\frac{143}{3}$ **(E)** $\frac{146}{3}$

21. 100 students in the Arvine Unified School District are taking a quiz. Each student randomly submits a real number between 0 and 1. All students submit a different real number. A student's score is the minimum positive difference between his or her number and another student's number. Let M be the maximum number of distinct scores. What is the probability that there will be M distinct scores?

(A) $\frac{2^{98}}{100!}$ **(B)** $\frac{2^{99}}{100!}$ **(C)** $\frac{2^{98}}{99!}$ **(D)** $\frac{2^{99}}{99!}$ **(E)** $\frac{2^{98}}{98!}$

22. Let ABC be a triangle with $AB = 8$, $BC = 7$ and $CA = 5$. Let ω denote the circumcircle of ABC. The tangent to the ω at B meet line AC and the tangent at C to ω at points D and E, respectively. Let the circumcircle of ABE meet segment CD at F. What is the length of EF?

(A) $\frac{35}{8}$ **(B)** $\frac{7\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{645}}{5}$ **(D)** $\frac{5\sqrt{70}}{8}$ **(E)** $\frac{40}{7}$

23. Given that $x^4 + ax^3 + bx^2 + 4ax + 16$ has four distinct positive real roots for integers a and b, what is the smallest possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 23 **(E)** 26

24. For how many real numbers $1 \leq x \leq 20$ is $\lfloor x \rfloor \{\sin(\pi x^2)\}$ an integer? (Note: $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n and $\{n\} = n - \lfloor n \rfloor$ denotes the fractional part of n.)

(A) 10043 **(B)** 10044 **(C)** 10071 **(D)** 10261 **(E)** 11661

25. Define the function $F(a, b)$ for two relatively prime integers $a, b \geq 2$ as the smallest positive multiple of a that leaves a remainder of 1 upon division by b. If a and b are not relatively prime, $F(a, b) = \frac{ab}{2}$. Find $\sum_{a=2}^{15} \sum_{b=2}^{15} F(a, b)$.

(A) 7080.5 **(B)** 7090 **(C)** 7137.5 **(D)** 7200 **(E)** 14400

2021

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DO NOT OPEN UNTIL SATURDAY, January 23, 2021

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Correspondence about the problems and solutions
for this exam should be sent by email to:

ocmathcircle@gmail.com.

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent OMC 12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE January 23, 2021.

2. YOU must not verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on maa.org/amc under "AMC 10/12") that you followed all rules associated with the administration of the exam.

3. If you chose to obtain an AMC 10/12 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedEx or UPS is strongly recommended.

4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, World Wide Web, or digital media of any type during this period is a violation of the competition rules.

The **Orange Math Competitions**
are made possible by the contributions of
the following problem-writers, test-solvers,
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