# Discrete Round

October 24th, 2020

1. **[2]** On an 8-by-8 chessboard, Pranav wants the King to do some laps. To do this, he places the King on the bottom left corner and will move it one square to the right or one square up until the King reaches the upper right corner. Pranav then moves the King one square to the left or one square down until it comes back to the bottom left corner. This is a lap. If Pranav makes the King do 3 laps, what is the maximum number of different squares on the chessboard the King could have visited?

2. **[2]** How many positive integers at most 1000 can be expressed as the sum of 3 of its (not necessarily distinct) positive integer divisors?

3. **[3]** For how many pairs of integers $(m, n)$ with $1 \leq m < n \leq 100$ are there such that $m + n^2$ divides $mn^2 + 1$?

4. **[3]** Let $p = 257$, and $S = \{0, 1, \ldots, p-1\}$. Consider a function $f : \mathbb{Z} \to S$ satisfying the following properties

   - $f(x + p) = f(x)$ for all $x$.
   - $p$ divides $x + f(x^2) - f(x)$ for all $x \in S$ and $x \neq 1$.
   - $f(1) = 1$

   Find the value of $f(2)$.

5. **[5]** Kodvick needs to cite 10 links for his essay. His dad, Rick Astley, randomly changes 3 of the links to his famous masterpiece "Never Gonna Give You Up". Every minute Kodvick checks 5 links at random. Once he finds an incorrect link during his check, he corrects all the wrong links among those five and submits to Ms. Gu. As Ms. Gu opens all 10 links, the expected number of times she opens an incorrect link can be expressed as $m/n$, where $m$ and $n$ are relatively prime integers. Compute $m + n$.

6. **[5]** Kodvick is very bad at shuffling a standard deck of 52 cards. While shuffling, he accidentally flips a card over with probability $1/3$. After shuffling the cards, Kodvick attempts to get all the cards to face the same direction by going through the cards, one by one, in the following manner.

   - He does nothing to the first card.
   - If the current card is facing the same way as the previous one, he does nothing
   - Otherwise, he flips all the cards before the current card.

   Every time Kodvick flips $k$ cards, he loses $k$ brain cells. The expected number of brain cells that he will lose can be expressed as $m/n$, where $m$ and $n$ are relatively prime integers. Compute $m + n$.

7. **[6]** How many sequences of nine numbers $a_1$, $a_2$,..., $a_9$, each an integer from 1 to 12 inclusive, are there such that no two or three consecutive numbers sum to a multiple of 13?

8. **[7]** Find the sum of $\gcd(i, j) \gcd(j, k)$ over all triples of positive integers $(i, j, k)$ such that $ijk = 2020^2$.

9. **[8]** For a positive integer $n$, let $\theta(n)$ be the number of ways to write $n$ as $d_1 d_2 d_3 \cdots d_k$ where $d_1, d_2, \ldots, d_k$ are positive integers with $1 < d_1 \leq d_2 \leq d_3 \leq ... \leq d_k \leq 100$ and $k$ is an even positive integer. The sum

$$\sum_{i=2}^{\infty} \frac{\theta(n)}{n}$$

can be expressed as $m/n$, where $m$ and $n$ are relatively prime integers. Compute $m+n$.

10. **[9]** In the Kingdom of Orange, there are 7 cities, and each pair of cities is connected by a road. The mayor wants to close some roads down so that for each pair of cities, there still exists a route for one to travel from one city to the other. Define the *distance* as the least number of roads one must take. Surprisingly, there are no routes that use one more road than the *distance* for each pair of cities. How many possible sets of roads can the mayor close?