# Continuous Round

October 23rd, 2021

1. **[2]** Let $P(x)$ be a polynomial such that $P(\sin\theta) = cos(4\theta)$ for all $0 \leq \theta < 2\pi$. What is $P(0)$?

2. **[2]** Let $ABCD$ be an isosceles trapezoid with $AB = 20$, $CD = 30$ and $AD = BC = 13$. Let points $X$ and $Y$ lie on lines $DC$ and $BC$, such that $AXBY$ is a parallelogram. The perimeter of $AXBY$ can be written as $a + b\sqrt{c}$ for positive integers $a, b, c$. Find $a + b + c$.

3. **[3]** Let $f$ be a function defined on the set of positive reals that satisfies
$$f(x) = \sum_{n=1}^{\infty} f\left(\frac{x}{3^n}\right)$$
for all positive real numbers $x$. If $f(1) = 1$, what is the value of $f(27)$?

4. **[3]** Let $ABCD$ be a parallelogram, and let the angle bisector of angle $\angle ADC$ meet segment $BC$ at $X$. Suppose there is a circle $\omega$ that is tangent to all four sides of $ABXD$. Given $CD = 30$ and $\cos\angle ADC = -1/9$, find $AD$.

5. **[5]** There exists a region bounded by the graph of $y^4 - 2x^2y^2 - 3x^4 - 4y^2 + 12x^2 = 0$ for $y \geq 0$. Find the least integer greater than its area.

6. **[5]** The hyperbola $xy = 1$ is rotated by $60^\circ$ counterclockwise about the origin to a new hyperbola. If the sum of the coordinates of the intersection of the two hyperbolas that lies in the first quadrant can be written as $\sqrt{n}$ for some positive integer $n$, find $n$.

7. **[6]** Let $ABC$ be an obtuse isosceles triangle with $AB = BC$. Point $D$ is constructed on the extension of side $BC$ past $B$ such that $AD = AB$. Let $\omega$ denote the incircle of $ABC$. Suppose $\omega$ touches $AB$ and $BC$ at $Z$ and $X$, respectively. Given that $AB = 14$ and line $ZX$ bisects segment $AD$, the length $AC$ can be written as $\frac{m}{n}$ for relatively prime positive integers $m$ and $n$. What is $m + n$?

8. **[7]** Let $ABC$ be a triangle such that $AC > BC > AB$. Let $X$ be the center of circle $\omega$, drawn such that $\omega$ is tangent to $AC$ and the circumcircle of triangle $ABC$. Suppose $X$ lies on segment $BC$. Given that $BC - AB = 8$ and $\sin C = 3/5$, the radius of $\omega$ can be expressed as $\frac{m}{n}$ for positive relatively prime integers $m$ and $n$. Find $m + n$.

9. **[8]** On the complex plane, let $A$ be the area of the region bounded by points of the form $\frac{4}{3+r-ri}$ for some real number $r$. Compute $\lfloor 100A \rfloor$.

10. **[9]** Let $\Gamma_1$ and $\Gamma_2$ be two externally tangent circles, with centers $A$ and $B$. Points $P$ and $Q$ are selected on $\Gamma_1$ and $\Gamma_2$, respectively, such that $\angle APQ = \angle BQP$. Furthermore, the radii of $\Gamma_1$ and $\Gamma_2$ are 3 and 4 respectively and $\sin\angle ABQ = 24/25$. Given that the length of $PQ$ can be written as $\frac{m}{n}$ for relatively prime positive integers $m$ and $n$, what is $m + n$?