OAA OMC
Orange Math Competitions

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OMC 10

Orange Mathematics Competitions
Saturday, October 30, 2021

INSTRUCTIONS

1. DO NOT LOOK AT THE PROBLEMS UNTIL YOU ARE READY TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/` and mark you answer to each problem on the AMC 10 Answer Sheet with a number 2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the OMC, **you must submit your answers using the Submission Form found at `https://tinyurl.com/omc10submission`. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only pencils, erasers, rulers, and scratch paper are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10/12 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/`. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, sign your name in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-1012/`.
9. Enjoy the problems!

The Committee on the Orange Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

1. Jonus is required to take a pill at least once every two consecutive days. What is the least number of times he could have taken the pill in a week?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

2. Annie is playing a game with 3 red and 3 green marbles in a container. Every second, she selects two marbles from the container at random without replacement. She continues to draw pairs of marbles until the two selected are of different colors, in which case she wins. What is the least number of marbles that Annie must draw to guarantee a win?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** Annie is not guaranteed to win

3. What value of x satisfies
   $$\frac{2}{x} + \frac{19}{20} = \frac{20}{21}?$$

   (A) 190 **(B)** 210 **(C)** 420 **(D)** 760 **(E)** 840

4. Elysia and her three cousins are sharing fish tacos. If all four people ate a distinct composite number of fish tacos, what is the least number of fish tacos they could have eaten in total? (Note: 1 is not considered to be composite.)

   (A) 20 **(B)** 23 **(C)** 25 **(D)** 27 **(E)** 28

5. Rounded to the nearest percent, Joshua got a 44% on his true/false test. Given that the test contained at most 10 questions, how many questions were on the test?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

6. For which of the following pairs (a, b) is $a^2 + b = b^2 + a$?

   (A) $(-2022, 2021)$ **(B)** $(-2022, 2023)$ **(C)** $(-2021, 2020)$ **(D)** $(2021, -2022)$
   (E) $(2021, -2021)$

7. There are 10 integers written on a blackboard with mean n for some positive integer n. When a positive integer k is added on the list, the mean of the eleven numbers is $2n - 1$. Which of the following numbers could be k?

   (A) 2027 **(B)** 2028 **(C)** 2029 **(D)** 2030 **(E)** 2031

8. Three real numbers a, b, c are in an arithmetic progression in that order. If $ab = 13$ and $bc = 17$, what is the value of b?

   (A) $\sqrt{13}$ **(B)** $\sqrt[4]{221}$ **(C)** $\sqrt{15}$ **(D)** $\sqrt{17}$ **(E)** $\sqrt{30}$

9. Five sisters, Ann, Emily, Lauren, Mia, and Sophia, have ages 4, 7, 9, 14, and 16 respectively. On one summer day, a group of sisters whose ages sum up to 30 went to watch a movie while the other sisters stayed at home. Which two sisters **must** have been together?

   (A) Ann and Emily **(B)** Ann and Lauren **(C)** Ann and Sophia
   (D) Emily and Lauren **(E)** Mia and Sophia

10. How many positive integers N less than or equal to 1000 are there such that 75% of N's divisors are multiples of 3?

    (A) 24 **(B)** 25 **(C)** 36 **(D)** 37 **(E)** 38

11. What is the perimeter of the quadrilateral formed by the tangent lines to the circle $x^2+y^2 = 5$ at points $(1, 2), (1, -2), (-1, 2), (-1, -2)$?

    (A) $5\sqrt{5}$ **(B)** 15 **(C)** 20 **(D)** $10\sqrt{5}$ **(E)** 25

12. Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 3$. A point P on segment AB satisfies $\angle DPC = 90^\circ$ and $AP < BP$. What is the length of AP?

    (A) 1 **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\sqrt{5}$ **(E)** 3

13. For how many even positive integers n does the number 18^{18} leave a remainder of $\frac{n}{2}$ when divided by n?

    (A) 0 **(B)** 19 **(C)** 37 **(D)** 361 **(E)** 703

14. A cubic polynomial $P(x) = ax^3 + bx + c$ with $a \neq 0$ has three roots r, s, and t. Which of the following polynomials has roots $r + s$, $s + t$, and $t + r$?

    (A) $ax^3 - bx - c$ **(B)** $ax^3 - bx + c$ **(C)** $ax^3 + bx - c$ **(D)** $ax^3 + bx^2 - c$
    (E) $ax^3 + bx^2 + c$

15. Points A, B, C, and D are selected on a circle in that order such that $AB = BC = x$ and $CD = DA = y$ for positive integers x and y. If quadrilateral $ABCD$ has area 2025, how many possible ordered pairs (x, y) are there?

    (A) 3 **(B)** 9 **(C)** 15 **(D)** 21 **(E)** 27

16. Two right circular cones have vertices facing down. The first cone has a radius of 1 and a height of 7 and the second cone has a radius of 4 and a height of 4. Emily fills the first cone up with water until the very top. If she then pours water form the first cone to the second cone until both have the same height, what is this height?

    (A) $\frac{\sqrt[3]{4900}}{10}$ **(B)** $\sqrt[3]{5}$ **(C)** $\frac{7\sqrt[3]{20}}{10}$ **(D)** 2 **(E)** $\frac{7\sqrt{10}}{10}$

17. David has eight weights labeled 1 to 8. Weight number n weighs 2^n pounds for positive integers $1 \leq n \leq 8$. He randomly selects at least two of the weights and puts them on a scale. How many different scale readings David can get?

    (A) 247 **(B)** 248 **(C)** 255 **(D)** 256 **(E)** 502

18. Alex and his four friends share one whole pizza. Each person in the friend group ate twice as much or half as much as another friend. After finishing the pizza, Alex lists out the fractions of the pizza that the five people ate and calculates the median. What is the positive difference between the largest and smallest median that Alex could've gotten?

    (A) $\frac{15}{124}$ **(B)** $\frac{9}{68}$ **(C)** $\frac{64}{465}$ **(D)** $\frac{5}{36}$ **(E)** $\frac{7}{45}$

19. Let $k > 1$ be the smallest positive integer such that the sum of the first 2020 positive integers is a divisor of the sum of the first $2020k$ positive integers. What is the sum of the digits of k?

    (A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

20. For how many two-digit positive integers a is the quantity

    $$\left\lfloor \frac{a}{3^c} \right\rfloor$$

    always even for any nonnegative integer c? (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

21. Let $ABCD$ be a trapezoid with AB parallel to CD, $BC = 6$, $DA = 5$, and $AB < CD$. The angle bisectors of angles DAB and ABC intersect at a point X on segment CD. If $AB = AX$, what is the perimeter of trapezoid $ABCD$?

    (A) 25 **(B)** 26 **(C)** 27 **(D)** 28 **(E)** 29

22. Sushanth selects 5 distinct letters from the 21 letter string $TST\cdots ST$ where T and S are alternating. He takes the 5 letters and puts them in the order they were in in the original string. How many ways could he select the 5 letters such that the string he obtains is $TSTST$?

    (A) 792 **(B)** 1024 **(C)** 1287 **(D)** 1365 **(E)** 2002

23. Define a positive integer n to be k-*special* if the base k representation of n is a palindrome. How many positive integers less than or equal to 2021 are both 2-*special* and 4-*special*?

    (A) 19 **(B)** 20 **(C)** 22 **(D)** 24 **(E)** 37

24. Line segment $\overline{AB}$ is a diameter of a circle with $AB = 2$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the incenter of $\triangle ABC$ traces out a closed curve missing two points. What is the area of the region bounded by this curve?

    (A) $\pi(3-2\sqrt{2})$ **(B)** $2\sqrt{2}-2$ **(C)** 1 **(D)** $\pi-2$ **(E)** $\pi(\sqrt{2}-1)$

25. Nolan has eight cards and eight containers, each numbered 1 to 8. He places the card labelled $i+1$ in container i for $1 \leq i \leq 7$, and finally places card 8 in container 1. Suppose a card i is *special* if it is in container i for $1 \leq i \leq 8$. In every move, Nolan selects any two containers at random and switches the cards in them if the total number of *special* cards would strictly increase. Otherwise, he does nothing. What is the expected number of moves until all eight cards are *special*?

    (A) 49 **(B)** $\frac{621}{10}$ **(C)** 64 **(D)** $\frac{363}{5}$ **(E)** $\frac{761}{10}$

2021

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DO NOT OPEN UNTIL SATURDAY, OCtober 30, 2021

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Correspondence about the problems and solutions
for this exam should be sent by email to:

ocmathcircle@gmail.com.

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent OMC 10 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE October 30, 2021.

2. YOU must not verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on maa.org/amc under "AMC 10/12") that you followed all rules associated with the administration of the exam.

3. If you chose to obtain an AMC 10/12 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedEx or UPS is strongly recommended.

4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, World Wide Web, or digital media of any type during this period is a violation of the competition rules.

The **Orange Math Competitions**
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