OAA OMC
Orange Math Competitions

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OMC 12

Orange Mathematics Competitions
Saturday, October 30, 2021

INSTRUCTIONS

1. DO NOT LOOK AT THE PROBLEMS UNTIL YOU ARE READY TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem however you want. If you would like to create a more realistic test experience, then you may obtain an AMC 10 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/` and mark you answer to each problem on the AMC 10 Answer Sheet with a number 2 pencil. To simulate the real test, check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded in a real test. For the OMC, **you must submit your answers using the Submission Form found at** `https://tinyurl.com/omc10submission`**. Only answers submitted to the Submission Form will be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only pencils, erasers, rulers, and scratch paper are allowed as aids. No calculators, smartwatches, phones, computing devices, or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, you will ask yourself to record certain information on the answer form if you chose to obtain an AMC 10/12 Answer Sheet from `https://www.maa.org/math-competitions/amc-1012/`. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, sign your name in the space provided at the top of the Answer Sheet should you choose to obtain one from `https://www.maa.org/math-competitions/amc-1012/`.
9. Enjoy the problems!

The Committee on the Orange Math Competitions reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

1. What value of x satisfies
$$\frac{2}{x}+\frac{19}{20}=\frac{20}{21}?$$

(A) 190 **(B)** 210 **(C)** 420 **(D)** 760 **(E)** 840

2. In a room of 20 people, 65% are wearing a jacket and 40% are wearing a hat. If 4 people are wearing neither, how many people are wearing both?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

3. A magician has a bag of marbles. Every second, he either changes a single marble into two, or changes two marbles into four. If he started with 10 marbles and ended with 37 after 20 seconds, how many times did he change a single marble into two?

(A) 3 **(B)** 7 **(C)** 13 **(D)** 14 **(E)** 17

4. Rounded to the nearest percent, Joshua got a 44% on his true/false test. Given that the test contained at most 10 questions, how many questions were on the test?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

5. Let $f(x)$ be a rational function such that $f(\tan(x))=\tan(3x)$ for all real numbers x. What is the value of $f(1)$?

(A) -1 **(B)** $-\frac{\sqrt{2}}{2}$ **(C)** 0 **(D)** $\frac{\sqrt{2}}{2}$ **(E)** 1

6. Five sisters, Ann, Emily, Lauren, Mia, and Sophia, have ages 4, 7, 9, 14, and 16 respectively. On one summer day, a group of sisters whose ages sum up to 30 went to watch a movie while the other sisters stayed at home. Which two sisters **must** have been together?

(A) Ann and Emily **(B)** Ann and Lauren **(C)** Ann and Sophia
(D) Emily and Lauren **(E)** Mia and Sophia

7. What is the smallest positive integer value of n such that 10^n-1 is divisible by 81?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 27 **(E)** 81

8. How many positive integers N less than or equal to 1000 are there such that 75% of N's divisors are multiples of 3?

(A) 24 **(B)** 25 **(C)** 36 **(D)** 37 **(E)** 38

9. Let $ABCD$ be a rectangle with $AB=10$ and $BC=3$. A point P on segment AB satisfies $\angle DPC=90^\circ$ and $AP<BP$. What is the length of AP?

(A) 1 **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\sqrt{5}$ **(E)** 3

10. A cubic polynomial $P(x)=ax^3+bx+c$ with $a\neq 0$ has three roots r, s, and t. Which of the following polynomials has roots $r+s, s+t$, and $t+r$?

(A) ax^3-bx-c **(B)** ax^3-bx+c **(C)** ax^3+bx-c **(D)** ax^3+bx^2-c
(E) ax^3+bx^2+c

11. Two right circular cones have vertices facing down. The first cone has a radius of 1 and a height of 7 and the second cone has a radius of 4 and a height of 4. Emily fills the first cone up with water until the very top. If she then pours water form the first cone to the second cone until both have the same height, what is this height?

(A) $\frac{\sqrt[3]{4900}}{10}$ **(B)** $\sqrt[3]{5}$ **(C)** $\frac{7\sqrt[3]{20}}{10}$ **(D)** 2 **(E)** $\frac{7\sqrt{10}}{10}$

12. Define a sequence a to majorize a sequence b where a_i and b_i are defined for $1 \leq i \leq n$ if

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \cdots + a_m \geq \sum_{k=1}^{m} b_k = b_1 + b_2 + \cdots + b_m$$

for all $1 \leq m \leq n$ with equality when $m = n$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ be a permutation of $(1, 2, 3, 4, 5, 6)$ such that σ majorizes $(1, 2, 3, 4, 6, 5)$. How many such permutations σ are there?

(A) 120 **(B)** 240 **(C)** 360 **(D)** 480 **(E)** 600

13. Suppose a polynomial $P(x) = x^3 + 3x + 4$ has roots r_1, r_2, and r_3. What is the value of

$$\frac{1}{r_1^2 + 3} + \frac{1}{r_2^2 + 3} + \frac{1}{r_3^2 + 3}?$$

(A) -12 **(B)** -4 **(C)** 0 **(D)** 4 **(E)** 12

14. David has eight weights labeled 1 to 8. Weight number n weighs 2^n pounds for positive integers $1 \leq n \leq 8$. He randomly selects at least two of the weights and puts them on a scale. How many different scale readings David can get?

(A) 247 **(B)** 248 **(C)** 255 **(D)** 256 **(E)** 502

15. Let triangle ABC be a equilateral triangle with side length 20. Let P be a point on minor arc $\widehat{BC}$ of the circumcircle of ABC. Given that $AP = 22$, $\max(BP, CP)$ can be expressed as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

(A) 22 **(B)** 31 **(C)** 33 **(D)** 48 **(E)** 60

16. Let $k > 1$ be the smallest positive integer such that the sum of the first 2020 positive integers is a divisor of the sum of the first $2020k$ positive integers. What is the sum of the digits of k?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

17. For how many two-digit positive integers a is the quantity

$$\left\lfloor \frac{a}{3^c} \right\rfloor$$

always even for any nonnegative integer c? (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

18. A frog is in a field bounded by $y = -0.5, y = 10.5, x = -0.5$, and $x = 10.5$. Every second, if the frog is at (a, b), it hops to any lattice point, with equal chance, inside the field on either the line $x = a$ or $y = b$. Note that the frog is allowed to stay put during a move. If the frog starts at the origin, find the expected number of seconds until the frog reaches $(10, 10)$.

(A) 70 **(B)** 99 **(C)** 100 **(D)** 114 **(E)** 121

19. Let $ABCD$ be a trapezoid with AB parallel to CD, $BC = 6$, $DA = 5$, and $AB < CD$. The angle bisectors of angles DAB and ABC intersect at a point X on segment CD. If $AB = AX$, what is the perimeter of trapezoid $ABCD$?

(A) 25 **(B)** 26 **(C)** 27 **(D)** 28 **(E)** 29

20. Let p_n denote the nth prime. Given that there is a real number γ such that

$$p_n = \lfloor 10^{2^n} \gamma \rfloor - 10^{2^{n-1}} \lfloor 10^{2^{n-1}} \gamma \rfloor,$$

what is the sum of the first 5 nonzero digits in the decimal expansion of γ? (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

(A) 17 **(B)** 18 **(C)** 22 **(D)** 27 **(E)** 28

21. Sushanth selects 5 distinct letters from the 21 letter string $TST \cdots ST$ where T and S are alternating. He takes the 5 letters and puts them in the order they were in in the original string. How many ways could he select the 5 letters such that the string he obtains is $TSTST$?

(A) 792 **(B)** 1024 **(C)** 1287 **(D)** 1365 **(E)** 2002

22. Let ABC be a triangle with $BC = a$ and $AC = b$ where a and b are positive integers. All three vertices of ABC lie on a circle with diameter $\sqrt{a^2 + b^2}$. What is the smallest possible value of $a + b$ given that the product of all possible values of AB is equal to 2020?

(A) 121 **(B)** 202 **(C)** 212 **(D)** 1010 **(E)** 2340

23. Define $f(k)$ to be the largest integer b such that 3^b divides $k!$. Let n be the 1000^{th} smallest positive integer n such that $n = 2f(n) + 1000$. What are the last three digits of n?

(A) 000 **(B)** 150 **(C)** 326 **(D)** 332 **(E)** 442

24. Nolan has eight cards and eight containers, each numbered 1 to 8. He places the card labelled $i+1$ in container i for $1 \le i \le 7$, and finally places card 8 in container 1. Suppose a card i is *special* if it is in container i for $1 \le i \le 8$. In every move, Nolan selects any two containers at random and switches the cards in them if the total number of *special* cards would strictly increase. Otherwise, he does nothing. What is the expected number of moves until all eight cards are *special*?

(A) 49 **(B)** $\frac{621}{10}$ **(C)** 64 **(D)** $\frac{363}{5}$ **(E)** $\frac{761}{10}$

25. Let $p = 157$ be a prime. How many triples of positive integers (x, y, z) are there such that $x + y + z = p$ and p divides $xy + yz + zx$?

(A) 78 **(B)** 79 **(C)** 156 **(D)** 157 **(E)** 313

2021

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DO NOT OPEN UNTIL SATURDAY, OCtober 30, 2021

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Correspondence about the problems and solutions for this exam should be sent by email to:

ocmathcircle@gmail.com.

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent OMC 12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE October 30, 2021.
2. YOU must not verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on maa.org/amc under "AMC 10/12") that you followed all rules associated with the administration of the exam.
3. If you chose to obtain an AMC 10/12 Answer Sheet from the MAA's website, it must be returned to yourself the day after the competition. Ship with inappropriate postage without using a tracking method. FedEx or UPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, World Wide Web, or digital media of any type during this period is a violation of the competition rules.

The **Orange Math Competitions** *are made possible by the contributions of the following problem-writers, test-solvers, and event coordinators:*

Sabreen Alam
Andrew Du
Andrew Gu
Won Jang
Min Kim
Sushanth S. Kumar
Emily Zhang
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