Discrete Round

October 23rd, 2021

1. **[2]** Sushanth would like to paint his candy corn to be more unique. He does this by choosing three different colors from the following options: red, orange, yellow, green, blue, purple, and white. How many ways can Sushanth choose the three colors so that his candy corn has a different combination of colors than a normal candy corn?

2. **[2]** Call a positive integer *scary* if none of its positive proper divisors are *scary*. Given that 1 is not scary, how many *scary* integers are there from 1 to 100, inclusive?

3. **[3]** Let $f(x)$ be the function that takes in a terminating real decimal and outputs its last nonzero digit. For example, $f(22.37) = 7$ and $f(400) = 4$. How many positive integers $1 \leq x \leq 2021$ are there such that $f(x) = f\left(\frac{x}{4}\right)$?

4. **[3]** Dob stands in the middle of a gigantic neighborhood with n houses. Interestingly, the n houses are situated in a circle centered around Dob so that they are equally spaced. Bobby starts at an arbitrary house for trick-or-treating. Once in a while, Bobby gets bored, so he wants to go to one of the farthest houses he can run in a straight line to. In particular, Dob must not be in the way of Bobby. For how many positive integers n from 3 to 2021 is it possible for Bobby to reach all n houses at some point?

5. **[5]** If p and q are prime numbers such that p divides $4q - 1$ and q divides $4p - 1$, find the smallest possible value of $p + q$.

6. **[5]** Jerethy randomly chooses 5 points on a 3-by-3 grid to be red. He then draws a segment between any two red points that doesn't pass through any other red points. If the expected number of segments that Jerethy draws can be expressed as $\frac{m}{n}$ in simplest form, what is $m + n$?

7. **[6]** An arithmetic sequence $a_1, a_2, a_3, \ldots$ is said to be *Kodvick-spooky* if the following list,

$$a_1, a_2 + a_3, a_4 + a_5 + a_6, a_7 + a_8 + a_9 + a_{10}, \ldots$$

is an increasing sequence of consecutive perfect cubes. Find the sum of all possible values of $\sum_{k=1}^{100} a_k$ over all possible Kodvick-spooky sequences.

8. **[7]** Find the largest positive integer $n \geq 2$ such that $n! + 861$ is a integer multiple of $n^2 - 1$.

9. **[8]** Kodvick is playing with a standard 3×3 grid in Terraria. Each second a random 3×1 or 1×3 section of the grid is hit by a meteor fragment. If the expected number of seconds it takes for all squares in the grid to be hit by a meteor fragment can be written as $\frac{m}{n}$ in simplest form, find $m + n$?

10. **[9]** Jimmy is playing a game with his Halloween candy. Initially, he keeps his 2016 pieces of candy in a single pile and he has a score of 0. Then, he makes a series of moves, where each move consists of taking one pile of candy and splitting it into two smaller piles. If there were $a + b$ pieces of candy in the original pile and he splits it into a pieces of candy and b pieces of candy, then his score increases by ab. However, there is one additional rule he must abide by: no two piles may have the same amount of candy at any time. Given this, what is the highest possible score Jimmy can get?