OMC 8 Solutions Manual

STEAM for All

17 October 2020

1 Answer Key

1. A
2. A
3. B
4. D
5. E
6. C
7. E
8. B
9. C
10. B
11. E
12. B
13. C
14. D
15. E
16. C
17. D
18. C
19. A
20. E
21. C
22. E
23. A
24. A
25. C

2 Solutions

1. What is the value of

$$2^{-2+0+2+0} + 0^{2-0+2+0} + 2^{2+0-2+0} + 0^{2+0+2-0} \ ?$$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution:

$$2^{-2+0+2+0} + 0^{2-0+2+0} + 2^{2+0-2+0} + 0^{2+0+2-0} = 2^0 + 0^4 + 2^0 + 0^4 = 1 + 0 + 1 + 0 = 2. \rightarrow \boxed{\text{A}}$$

2. 200 students were surveyed about their favorite candy. 72% of the surveyed students said that Snickers bars are their favorite candy. If twice as many girls as boys stated that Snickers bars are their favorite candy, how many boys stated that Snickers bars are their favorite candy?

(A) 48 **(B)** 72 **(C)** 96 **(D)** 120 **(E)** 144

Solution: We know that $0.72(200) = 144$ students stated that Snickers bars are their favorite candy. If twice as many girls as boys stated that Snickers bars are their favorite candy, this means one-third of the students that said Snickers bars are their favorite candy were boys. Therefore our answer is $144 \cdot \frac{1}{3} = 48. \rightarrow \boxed{\text{A}}$

3. In a class of 35 students, 19 have brown eyes and 18 have black hair. If 10 have both brown eyes and black hair, how many students have neither brown eyes nor black hair?

(A) 6 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 27

Solution: If we add up the number of students that have brown eyes or black hair, we get $19 + 18 = 37$ students. However, we have overcounted the students that have both brown eyes and black hair, because they are added once for the brown eyes cateogry and another time for the black hair category. Hence, we subtract 10 to get that the number of students that have brown eyes or black hair is 27. Therefore $35 - 27 = 8$ students have neither brown eyes nor black hair. $\rightarrow \boxed{\text{B}}$

4. Neel rolls a fair 6-sided die and then rolls a fair 5-sided die. What is the probability that the two numbers that come up sum to 7?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{5}{36}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{5}$

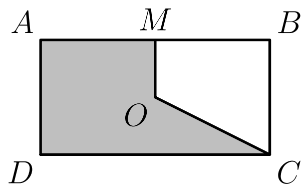
Solution: We can see that there is a total of $6 \cdot 5 = 30$ equally likely possible outcomes for the dice rolls. (2,5), (3,4), (4,3), (5,2), and (6,1) are all valid pairs (where the first number is from the 6-sided die and the second number is from the 5-sided die) so there are exactly 5 outcomes that have the two numbers sum to 7. Therefore our answer is $\frac{5}{30} = \frac{1}{6}. \rightarrow \boxed{\text{D}}$

5. Andrew eats a Twix bar every ten days in 2020. He eats his first Twix bar of the year on Wednesday, January 1, 2020. What day of the week will Andrew eat his last Twix bar of the year?

(A) Sunday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

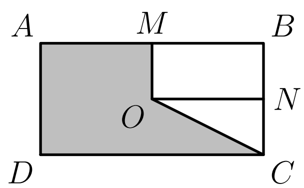
Solution: Andrew will eat a Twix bar on the 1st day of the year, the 11th day of the year, the 21st day of the year, and any day of the year with a units digit of 1. Note that there are 366 days in 2020 (it is a leap year). Thus, the last day Andrew eats a Twix bar is on day 361. Note that day 358 is a Wednesday because 357 is a multiple of 7. Therefore, day 361 must be on a Saturday. → $\boxed{\text{E}}$

6. Let $ABCD$ be a rectangle with $AB = 2$, $BC = 1$, and center O. Let M be the midpoint of AB. What is the area of the concave pentagon $AMOCD$?



(A) $\frac{9}{8}$ **(B)** $\frac{7}{6}$ **(C)** $\frac{5}{4}$ **(D)** $\frac{4}{3}$ **(E)** $\frac{11}{8}$

Solution: To solve this problem, we can find the area of $OMBC$ and subtract it from the area of $ABCD$. Notice that the area of $ABCD$ is $2 \cdot 1 = 2$. Now, to find the area of $OMBC$, we can divide it strategically into a rectangle and a triangle. Let N be the midpoint of BC, as shown in the diagram below.



Clearly, $OMBN$ is a rectangle with $OM = \frac{1}{2}$ and $MB = 1$, and ONC is a triangle with $NC = \frac{1}{2}$ and $ON = 1$. Hence, the area of $OMBC$ is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{3}{4}$. To find the area of $AMOCD$, we can simply subtract this value from the area of $ABCD$. Our final answer is $2 - \frac{3}{4} = \frac{5}{4}$. → $\boxed{\text{C}}$

7. The positive integer $\underline{2}\ \underline{0}\ \underline{2}\ \underline{0}\ \underline{X}\ \underline{Y}$ is divisible by 18. What is the largest possible value of $X - Y$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution: We know that in order for a positive integer to be divisible by 18, it must be divisible by 2 and 9. From the divisibility condition for 2, we know that the units digit, Y, must be even. From the divisibility condition for 9, we know that the sum of the digits must be a multiple of 9 so $2+0+2+0+X+Y = 9$ or $2+0+2+0+X+Y = 18$ (any other multiples of 9 would make $X + Y$ too big for X and Y to be digits). The first case gives us $X + Y = 5$. To maximize $X - Y$, we must maximize X and minimize Y. Thus, if we set $X = 5$ and $Y = 0$ we can see that this does indeed satisfy Y being even so $5 - 0 = 5$ is attainable. For the second case, we know $X + Y = 14$. To maximize $X - Y$, we can set $X = 9$ and $Y = 5$. Unfortunately, Y is odd in this case, so the next best case is when $X = 8$ and $Y = 6$. This does indeed work, so $8 - 6 = 2$ is the largest possible in this case. Combining the two cases we see that 5 is the largest possible value of $X - Y$. → $\boxed{\text{E}}$

8. What is the 100th letter in the sequence $ABBBCCCCCDDDDDDD\cdots$ where the n^{th} letter in the alphabet comes up $2n - 1$ times?

(A) I **(B)** J **(C)** K **(D)** L **(E)** M

Solution: Let us consider each string of the same letters as a block. Notice that the number of letters in the sequence after k blocks is equal to $1 + 3 + 5 + \cdots + 2k - 1$. It is well-known that the sum of the k consecutive odd integers is equal to k^2. (If you've never seen this before, I suggest that you try to prove it on your own!) Because $100 = 10^2$, the 100^{th} letter is at the last letter of the 10^{th} block. Therefore, our answer is the 10^{th} letter of the alphabet which is J. $\rightarrow \boxed{\text{B}}$

9. Ian wants to buy exactly 25 jack-o-lanterns at a store for his Halloween party. He knows of three brands that sell jack-o-lanterns. The first brand sells packs of 5 jack-o-lanterns for \$4. The second brand sells packs of 7 jack-o-lanterns for \$6. The third brand sells packs of 11 jack-o-lanterns for \$7. What is the least amount of money Ian can spend to buy his jack-o-lanterns?

   (A) \$17 **(B)** \$18 **(C)** \$19 **(D)** \$20 **(E)** \$21

   Solution: If we test possible cases, we can see that there are exactly two possibilities: Five packs from the first brand or two packs from the second brand and one pack from the third brand. The first possibility costs $5 \cdot \$4 = \$20$. The second possibility costs $2 \cdot \$6 + \$7 = \$19$. Therefore, the least amount of money Ian can spend is \$19. $\rightarrow \boxed{\text{C}}$

10. Joshua has a box of candies. First, he gives nine less than half of his candies to his older sister. Then, he gives two more than one-third of his remaining candies to his best friend Ben. Finally, he eats two less than half of his remaining candies, leaving him with exactly fifteen left. How many candies did he have in the beginning?

    (A) 54 **(B)** 66 **(C)** 90 **(D)** 114 **(E)** 162

    Solution: Let x be the number of candies Joshua started with. If he gives 9 less than half of his candies to his older sister, that means he has $\frac{x}{2} + 9$ candies remaining. Then, he gives two more than one-third of his remaining candies to Ben. This means he has two less than two-thirds of his remaining candies left, which is equal to $\frac{x}{3} + 6 - 2 = \frac{x}{3} + 4$. Finally, he eats two less than half of his remaining candies, so he has two more than half of his remaining candies left. Two more than half of his remaining candies is equal to $\frac{x}{6} + 2 + 2 = \frac{x}{6} + 4$. This value must be equal to 15. Equating we get $\frac{x}{6} + 4 = 15$, so $x = 66$. $\rightarrow \boxed{\text{B}}$

11. Let $a \neq b$ be real non-zero numbers that satisfy $a^2 + 5b = b^2 + 5a$. What is the value of $a + b$?

    (A) -5 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 5

    Solution We can rearrange the given equation to get $a^2 - b^2 = 5a - 5b$. Using our difference of squares factorization pattern and taking out a common factor of 5 from the right hand side, the equation becomes $(a+b)(a-b) = 5(a-b)$. Now, since $a \neq b$, we know $a - b \neq 0$, so we can divide by $a - b$ on both sides. This tells us $a + b = 5$. $\rightarrow \boxed{\text{E}}$

12. A ghost is on the number line. Every second, the ghost randomly chooses between gliding one unit in the positive direction or two units in the negative direction. What is the probability that after 6 seconds, the ghost is at its original position?

    (A) $\frac{5}{32}$ **(B)** $\frac{15}{64}$ **(C)** $\frac{5}{16}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{15}{32}$

    Solution: In order for the ghost to be at its original position, it must have glided one unit in the positive direction four times and two units in the negative direction two times. Therefore, the number of ways the ghost can be at its original position is equal to the number of ways to arrange the letters $PPPPNN$. This is just equal to $\frac{6!}{4!2!} = 15$. The total number of possibilities is equal to 2^6 because for each second, there are two different situations that occur equally at random. Therefore our answer is $\frac{15}{64}$. $\rightarrow \boxed{\text{B}}$

13. How many of the 2021 positive divisors of 2^{2020} leave a remainder of 2 when divided by 3?

(A) 673 **(B)** 674 **(C)** 1010 **(D)** 1011 **(E)** 1347

Solution: Notice that all divisors of 2^{2020} are of the form 2^n for some integer n between 0 and 2020 inclusive. Thus, we are searching for the number of integers n between 0 and 2020 inclusive such that $2^n \equiv 2 \pmod 3$. Notice, however, that $2^2 \equiv 1 \pmod 3$. Therefore, for any integer k, we know $2^{2k} \equiv (2^2)^k \equiv 1^k \equiv 1 \pmod 3$. Now, we see that if n is even, we must have $2^n \equiv 1 \pmod 3$. If n is odd, we know that $n = 2k+1$ for some integer k. Then $2^n \equiv 2^{2k+1} \equiv 2 \cdot 2^{2k} \equiv 2 \pmod 3$. Thus, for all odd integers n, we know $2^n \equiv 2 \pmod 3$.

The problem reduces itself to finding the number of odd integers between 0 and 2020 inclusive. The answer is 1010. $\rightarrow \boxed{\text{C}}$

14. Let f be a real valued function such that

$$2f(x) + 3f\left(\frac{10}{x}\right) = 5x$$

for all real $x \neq 0$. What is the value of $f(2)$?

(A) -4 **(B)** 0 **(C)** 11 **(D)** 14 **(E)** 28

Solution: Let $y = f(2)$ and $z = f(5)$. Plugging in $x = 2$ to the functional equation, we find $2f(2) + 3f(5) = 10$. Plugging in $x = 5$, we find $2f(5) + 3f(2) = 25$. Now, we have the system of equations

$$2y + 3z = 10,$$
$$3y + 2z = 25.$$

Solving using your method of choice, we find $y = f(2) = 11$. $\rightarrow \boxed{\text{C}}$

15. Kelly is going to a campsite 90 miles away from her home. For the first 60 miles, she takes a train that travels at an average speed of 40 miles per hour. For the remainder of the trip, Kelly takes a taxi. If the average speed over the entire trip is 48 miles per hour, what is the average speed of the taxi, in miles per hour?

(A) 52 **(B)** 56 **(C)** 60 **(D)** 64 **(E)** 80

Solution: First, recall $d = rt$ where d is distance, r is rate, and t is time. In other words, we know distance is rate multiplied by time. Let t_k be the total time (in hours) of Kelly's trip. Since her average speed (her rate) is 48 miles per hour, the total distance she travels can be expressed as $48t_k$. However, Kelly travels 90 miles, so $48t_k = 90$ and $t_k = \frac{90}{48} = \frac{15}{8}$ hours.

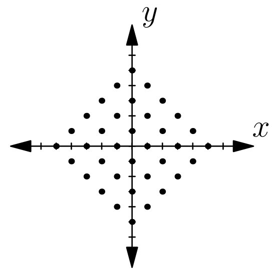
Rearranging our equation $d = rt$ yields $t = \frac{d}{r}$. This means the total time she spends on the train is $\frac{60}{40} = \frac{3}{2}$ hours. Therefore, Kelly spends $\frac{15}{8} - \frac{3}{2} = \frac{3}{8}$ hours on the taxi and goes $90 - 60 = 30$ miles.

Rearranging our equation $d = rt$ one more time gives $r = \frac{d}{t}$. To calculate Kelly's average speed on the taxi, we can simply plug in $d = 30$ and $t = \frac{3}{8}$. Our answer is $\frac{30}{\frac{3}{8}} = 30\left(\frac{8}{3}\right) = 80$ miles per hour. $\rightarrow \boxed{\text{E}}$

16. Every second, Sushanth the ant randomly chooses between crawling one unit up, right, down, or left in the coordinate plane. If Sushanth starts at the origin, how many different locations could he be after five seconds?

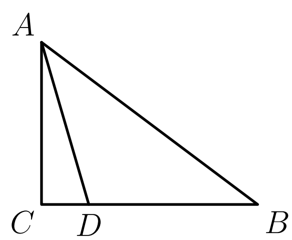
(A) 20 **(B)** 30 **(C)** 36 **(D)** 48 **(E)** 60

Solution:



We can consider Sushanth's five moves to be a string of U, R, D, and L where these stand for up, right, down, and left respectively. We can first eliminate pairs of U and D and pairs of R and L as these will cancel out. We see that the number of letters eliminated come in pairs, so the total number of disregarded moves is even. Since Sushanth makes 5 moves and odd num − even num = odd num, the minimum number of moves to reach Sushanth's current position, starting from the origin, is odd. In other words, the Manhattan distance from Sushanth's final position to the origin must be an odd number. This means Sushanth can be 1 unit, 3 units, or 5 units away from the origin. Thus, there are a total of $4(1+3+5)=36$ points possible. $\rightarrow \boxed{\text{C}}$

17. Let ABC be a right triangle with $\angle C = 90°$, $AB = 5$, and $AC = 3$. Let D be a point on BC such that $AD = BD$. Find CD.



(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{7}{8}$ **(E)** 1

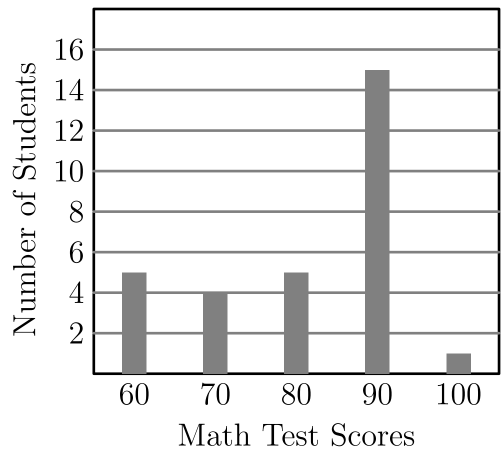
Solution: By the Pythagorean Theorem, $BC = \sqrt{5^2 - 3^2} = 4$. Let x be the length of CD. Then, $BD = BC - CD = 4 - x$. Because $AD = BD$, $AD = 4 - x$ as well. Now, applying the Pythagorean Theorem on $\triangle ACD$, we get $3^2 + x^2 = (4-x)^2$. Simplifying, we have $9 + x^2 = x^2 - 8x + 16$, so $8x = 7$. This means $x = \frac{7}{8}$. $\rightarrow \boxed{\text{D}}$

18. Gary, Katherine, and Michael are running laps around a circular track. It takes Gary, Katherine, and Michael 10, 20, and 30 seconds respectively to complete one lap. Every time one of them completes a lap, they add one to their team score. If they begin running with 0 points and at the starting line, what is the least number of seconds it takes for them to have a score of at least 100 points?

(A) 510 **(B)** 540 **(C)** 550 **(D)** 960 **(E)** 990

Solution: In 60 seconds, Gary runs 6 laps, Katherine runs 3 laps, Michael runs 2 laps, and they are all back at the starting line. Thus, their team score will increase by $6+3+2=11$ points every minute. This minutes that after 9 minutes, the team would have a total of $9 \cdot 11 = 99$ points. For the last point, Gary will have to finish a lap. Thus, the total time, in seconds, is equal to $9 \cdot 60 + 10 = 550$. $\rightarrow \boxed{\text{C}}$

19. A witch gives a 100 question math test to her students. All raw scores were positive integers greater than or equal to 55. She then rounds each score to the nearest ten. The scores are shown below in a bar graph.



Which of the following statements about the raw scores before the witch rounded **must** be true?

I: The median is greater than the mean.
II: The highest score is 100.
III: The range is equal to 40.
IV: The mode and the median are the same.

(A) I only **(B)** II and III only **(C)** II, III, and IV only **(D)** I, II, III, and IV **(E)** None of the statements must be true

Solution: We shall test each of the statements one by one.

I: Note that the median must lie between 85 and 94 inclusive. The largest possible value of the mean is

$$\frac{5 \cdot 64 + 4 \cdot 74 + 5 \cdot 84 + 15 \cdot 94 + 100}{30} = \frac{2546}{30} \approx 84.87 < 85.$$

Thus, the mean must be smaller than the median, so this statement is true.

II: The highest score does not necessarily have to be 100. It is only known that the highest score rounded to the nearest ten is equal to 100. A counterexample is if the maximum score was 98.

III: The range does not necessarily have to equal 40. The values in the chart are rounded to the nearest ten, and are not the raw scores. A counterexample is if the minimum score was 55 and the maximum score was 100, the range would be 45.

IV: Because it is the mode of the raw scores, the mode can almost be anything! Also, the only constraint for the median is that it lies between 85 and 94 inclusive. Thus, it is definitely possible for them to not be equal to each other.

Therefore, the only statement that must be true is I. $\rightarrow \boxed{\text{A}}$

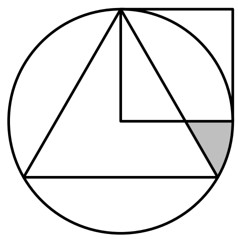
20. Alex is playing a game of hot potato with his three friends Bela, Christine, and Daisy. Every second, whoever is holding the potato randomly selects one of the other three players and throws him or her the potato. If Alex is currently holding the potato, what is the probability that after three seconds Alex is **not** holding the potato?

(A) $\frac{2}{3}$ **(B)** $\frac{19}{27}$ **(C)** $\frac{20}{27}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{9}$

Solution 1: We will solve this by finding the probability Alex is holding the potato after three seconds and subtracting it from 1. Note that after the first second, Alex must have thrown the potato to one of Bela, Christine, or Daisy. It does not matter who Alex threw the potato to, so without loss of generality, let us assume Alex threw it to Bela. If Bela throws Alex the potato in the next second, then it is impossible for Alex to have the potato after three seconds since he would have to throw it to someone else. Therefore, the only possible way Alex can get the potato after three seconds is if Bela throws it to one of Christine or Daisy and whoever gets the potato from Bela throws it to Alex. There is a $\frac{2}{3}$ probability Bela throws the potato to one of Christine or Daisy. Then, there is a $\frac{1}{3}$ probability that whoever catches the potato from Bela throws it to Alex. Therefore, our probability is equal to $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$. However, the probability we want is the complement of this, so our answer is $\frac{7}{9}$. $\rightarrow \boxed{\text{E}}$

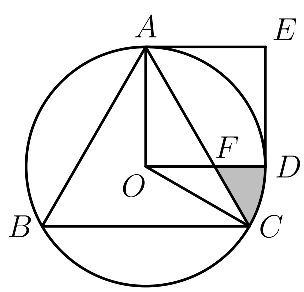
Solution 2: We will solve this using complementary probability. Let p_n denote the probability that Alex is holding the potato after n seconds. The only possible way Alex is holding the potato is if someone other than him threw him the potato with probability $\frac{1}{3}$. Therefore $p_n = \frac{1}{3}(1 - p_{n-1})$. Using $p_0 = 1$ we can find that $p_1 = 0$, $p_2 = \frac{1}{3}$, and $p_3 = \frac{2}{9}$. However, the probability we want is the complement of p_3, so our answer is $\frac{7}{9}$. $\rightarrow \boxed{\text{E}}$

21. In the figure below, an equilateral triangle is inscribed in a circle with radius 1. A square with one vertex as the center of the circle and an adjacent vertex as the vertex of the equilateral triangle is drawn. What is the area of the shaded region?



(A) $\frac{\sqrt{3}}{24}$ **(B)** $\frac{\pi}{12} - \frac{1}{6}$ **(C)** $\frac{\pi}{12} - \frac{\sqrt{3}}{12}$ **(D)** $\frac{\pi}{12} - \frac{\sqrt{3}}{18}$ **(E)** $\frac{\pi}{9} - \frac{\sqrt{3}}{12}$

Solution: Let the triangle have vertices A, B, C, the circle have center O, and the square have vertices A, O, D, E as shown below. Also, let F be the intersection of AC and OD.



Notice that the area of the shaded region can be expressed as the area of the sector with central angle $\angle COD$ minus the area of $\triangle OCF$. We first calculate the area of the sector with angle $\angle COD$. Because the central angle of an equilateral triangle is $120°$, we know $\angle AOC = 120°$. Also, $AODE$ is a square, so $\angle AOD = 90°$. Hence, $\angle COD = \angle AOC - \angle AOD = 30°$. Thus, the area of the sector with angle $\angle COD$ is $\frac{30°}{360°}(\pi) = \frac{\pi}{12}$.

To find the area of $\triangle OCF$ we first note $\angle COF = \angle COD = 30°$. Also, AO bisects $\angle BAC$, so $\angle OAC = 30°$. This means $\angle FCO = \angle ACO = 180° - \angle AOC - \angle OAC = 30°$. Finally, this means $\angle OFC = 180° - \angle FOC - \angle FCO = 120°$. Thus, we can think of $\triangle OFC$ as two 30-60-90 triangles pieced together. We know $AF : OF : AO = 2 : 1 : \sqrt{3}$ (this comes from our 30-60-90 triangle knowledge). Since the circle is a unit circle, we know $AO = 1$. Thus, $OF = \frac{1}{\sqrt{3}}$ since $\triangle AOF$ is a 30-60-90 triangle. We can now compute the height from F to OC to be $\frac{1}{2\sqrt{3}}$ and $OC = 1$ since it is the radius. This means the area of $\triangle OFC$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{12}$.

Putting this all together, we know the area of the shaded region is the area of the sector minus the area of $\triangle OFC$. This gives an answer of $\frac{\pi}{12} - \frac{\sqrt{3}}{12}$. $\rightarrow \boxed{\text{C}}$

22. Samuel is having a costume party at his house. He wants to invite 4 out of 8 of his friends. Two of his friends, Cynthia and Joyce, will only attend if the other is invited as well. Given that all of the people who were invited came to his costume party and that no one who was not invited came, what is the probability Cynthia and Joyce both went?

(A) $\frac{3}{14}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{7}$ **(E)** $\frac{1}{2}$

Solution: We can use the formula for conditional probability

$$P(A \mid B) = \frac{P(A \,\&\, B)}{P(B)},$$

where $P(A \mid B)$ denotes the probability A occurs given B occurs, and $P(A \,\&\, B)$ denotes the probability both A and B occur. In this problem, $P(A)$ is the probability Cynthia and Joyce went to the party (which means they were both invited), and $P(B)$ is the probability all the people invited came to Samuel's party.

There are a total of $\binom{8}{4} = 70$ ways Samuel can invite 4 of his 8 friends to the party. If he invites both Cynthia and Joyce, there are $\binom{6}{2} = 15$ ways Samuel can invite 2 other friends. Hence $P(A) = \frac{15}{70} = \frac{3}{14}$.

Now, we compute $P(B)$, the probability that all friends invited came to Samuel's party. This always occurs if neither Cynthia nor Joyce is invited, and also occurs if both Cynthia and Joyce are invited. If neither Cynthia nor Joyce is invited, Samuel must distribute the 4 invites among 6 friends, which he can do in $\binom{6}{4} = 15$ ways. Therefore, the probability neither Cynthia nor Joyce is invited is $\frac{15}{70} = \frac{3}{14}$. We calculated previously the probability of both Cynthia and Joyce being invited is also $\frac{15}{70} = \frac{3}{14}$. Hence $P(B) = 2 \cdot \frac{3}{14} = \frac{3}{7}$.

Using our formula for conditional probability, our answer is $\frac{\frac{3}{14}}{\frac{3}{7}} = \frac{1}{2}$. $\rightarrow \boxed{\text{E}}$

23. In a strictly increasing positive integer sequence, every term after the first two terms is the sum of the previous two terms in the sequence. Given that the eighth term is 2020, what is the maximum possible value of the first term?

(A) 90 **(B)** 96 **(C)** 100 **(D)** 246 **(E)** 252

Solution: Let the first term of the sequence be a and the second term of the sequence be b. Then we can recursively compute the eighth term of the sequence to be $8a + 13b$. We know $8a + 13b = 2020$ and $a < b$ from the problem statement. Now, notice that if we find a solution (a, b) another solution is $(a + 13, b - 8)$, because $8(a + 13) + 13(b - 8) = 8a + 104 + 13b - 104 = 2020$.

We can test a few values of a to find a possible solution. It's easy to confirm that $(a, b) = (12, 148)$ works. This satisfies $a < b$, but it doesn't give us the maximal value. To achieve it, we can repeatedly add 13 to a and subtract 8 from b. We find the max value of a occurs when $(a, b) = (90, 100)$. $\rightarrow \boxed{\text{A}}$

Remark: Another way to find the maximum possible value of a from $8a + 13b = 2020$ is to use modular arithmetic.

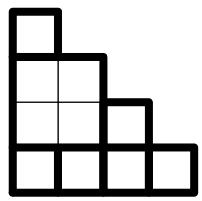
24. How many ordered triples of positive integers (x, y, z) are there such that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{2000}?$$

(A) 171 **(B)** 190 **(C)** 200 **(D)** 210 **(E)** 231

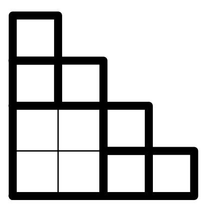
Solution: Note that $\sqrt{2000} = \sqrt{400 \cdot 5} = 20\sqrt{5}$. In order for the sum of three radicals to be a multiple of $\sqrt{5}$, each of those three radicals must also be a multiple of $\sqrt{5}$. Let $x = a\sqrt{5}$, $y = b\sqrt{5}$, and $z = c\sqrt{5}$. Then we have $a\sqrt{5} + b\sqrt{5} + c\sqrt{5} = 20\sqrt{5}$. Dividing by $\sqrt{5}$ on both sides gives us $a + b + c = 20$. Now, either by stars and bars or casework, we arrive at our answer of $\binom{19}{2} = 171$. $\rightarrow \boxed{\text{A}}$

25. A $n \times n$ staircase is defined to be a left-aligned triangular array of unit squares with one square in the first row, two squares in the second row, etc. What is the minimum number of squares of positive integer length that is needed to fill up a 32×32 staircase? For instance, the minimum number of squares needed to fill a 4×4 staircase is 7. A possible solution for a 4×4 staircase is shown below.



(A) 31 **(B)** 42 **(C)** 63 **(D)** 70 **(E)** 127

Solution: We can use a greedy-like algorithm where we place the largest square possible in the bottom left as shown below.



Notice that for any even n, the largest square we can place in the bottom left is a $\frac{1}{2}n \times \frac{1}{2}n$ square. We are then left with two different regions, one on top of the large square and one to the right. But notice that each of these is basically a smaller version of our original problem! Thus, if we let $f(2k)$ be the minimum number of squares for a $2k \times 2k$ staircase, $f(2k) = 1 + 2 \cdot f(k)$ where the 1 comes from the large square in the bottom left and the $f(k)$s come from the sections above and to the right of the large square. We know that $f(4) = 7$. This means that $f(8) = 1 + 2 \cdot 7 = 15$, $f(16) = 1 + 2 \cdot 15 = 31$, and $f(32) = 1 + 2 \cdot 31 = 63$. $\rightarrow \boxed{\text{C}}$